PARALLEL TRACTOR EXTENSION AND AMBIENT METRICS OF HOLONOMY SPLIT G_2

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1. Introduction

The work of Nurowski and Leistner-Nurowski in [N1], [N2], and [LN] has revealed a beautiful connection between the geometry of generic 2-plane fields $\mathcal{D}$ on manifolds M of dimension 5 and pseudo-Riemannian metrics of signature $(3,4)$ in dimension 7 whose holonomy is the split real form G_2 of the exceptional Lie group. (Throughout this paper, unqualified G_2 refers to the split real form. A 2-plane field on a 5-manifold is said to be generic if its second commutator spans TM at each point.) The starting point is Nurowski's observation in [N1] that using Cartan's solution [C] of the equivalence problem for such $\mathcal{D}$, one can invariantly associate to $\mathcal{D}$ a conformal class of metrics of signature $(2,3)$ on M. The ambient metric construction of [FG1] associates to a real-analytic conformal structure of signature (p,q) on an odd-dimensional manifold M a Ricci-flat metric $\widetilde{g}$ of signature $(p+1,q+1)$ on an open set $\widetilde{\mathcal{G}} \subset \mathbb{R}_+ \times M \times \mathbb{R}$ containing $\mathbb{R}_+ \times M \times \{0\}$. Specializing to Nurowski's conformal structures produces a metric of signature $(3,4)$ associated to each real-analytic $\mathcal{D}$.

In [N2], Nurowski identifies $\widetilde{g}$ explicitly for a particular 8-parameter family of generic 2-plane fields on $\mathbb{R}^5$. The family is parametrized by $\mathbb{R}^8$ via polynomial equations. This is remarkable in itself, as it is rare that $\widetilde{g}$ can be explicitly identified. Using Nurowski's formula for $\widetilde{g}$, Leistner-Nurowski show in [LN] that all the metrics in the family satisfy $\mathrm{Hol}(\widetilde{g}) \subset G_2$, and that if one of four of the eight parameters is nonzero, then $\mathrm{Hol}(\widetilde{g}) = G_2$. In particular, this gives a completely explicit 8-parameter family of metrics of holonomy G_2.

One reason this result is of interest is because metrics whose holonomy equals G_2 are not easy to come by. Despite their appearance on Berger's list in 1955, it was not until 1987 that Robert Bryant [Br1] first proved the existence of such metrics. The case of compact G_2 has received more attention in the intervening years, due partly to the role of manifolds of holonomy compact G_2 as an M-theory analogue of Calabi-Yau manifolds. But even in that case where more is known, new constructions are of interest.

The analysis of Leistner-Nurowski of the holonomy of these metrics depends crucially on Nurowski's explicit formula for $\widetilde{g}$. It is natural to ask whether analogous properties hold for more general $\mathcal{D}$. In this paper we show this is the case.

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For simplicity we take M to be oriented (which is equivalent to $\mathcal{D}$ being oriented). Our results extend easily to the non-orientable case. Our first main result extends the Leistner-Nurowski holonomy containment to general real-analytic $\mathcal{D}$.

Theorem 1.1. *Let $\mathcal{D} \subset TM$ be a generic 2-plane field on a connected, oriented 5-manifold M, with M and $\mathcal{D}$ real-analytic. Then* $\mathrm{Hol}(\widetilde{g}) \subset G_2$.

Our second main result provides sufficient conditions for $\mathrm{Hol}(\widetilde{g}) = G_2$. We impose two pointwise nondegeneracy conditions involving the Weyl and Cotton tensors of a representative g of Nurowski's conformal class. If $x \in M$, define the linear transformation $L_x : T_xM \times \mathbb{R} \to \otimes^3 T_x^*M$ by

$$L_x(v, \lambda) = W_{ijkl}v^i + C_{jkl}\lambda, \tag{1.1}$$

where W_{ijkl} denotes the Weyl tensor of g at x and C_{jkl} the Cotton tensor. The map L_x depends on the choice of representative g of the conformal class, but the conformal transformation laws of W and C show that its range, and therefore also its rank, are invariant under conformal rescaling. Our first condition is that L_x has rank 6, or equivalently, that it is injective.

Our second condition depends only on the Weyl tensor; in fact it depends only on the 5-dimensional piece of the Weyl tensor giving Cartan's basic curvature invariant A of generic 2-plane fields whose vanishing locally characterizes the homogeneous model. A is a section of $S^4\mathcal{D}^*$, i.e. it is a symmetric 4-form on $\mathcal{D}$. For $y \in M$, we say that A_y is 3-nondegenerate if the only vector $X \in \mathcal{D}_y$ satisfying $A(Y, X, X, X) = 0$ for all $Y \in \mathcal{D}_y$ is $X = 0$. We will say that A_y is 3-degenerate if it is not 3-nondegenerate.

Theorem 1.2. *Let $\mathcal{D} \subset TM$ be a generic 2-plane field on a connected, oriented 5-manifold M, with M and $\mathcal{D}$ real-analytic. Suppose there exist $x, y \in M$ such that L_x is injective and A_y is 3-nondegenerate. Then* $\mathrm{Hol}(\widetilde{g}) = G_2$.

In Theorems 1.1 and 1.2, the domain of $\widetilde{g}$ is taken to be a sufficiently small neighborhood of $\mathbb{R}_+ \times M \times \{0\}$ in $\mathbb{R}_+ \times M \times \mathbb{R}$ diffeomorphic to $\mathbb{R}_+ \times M \times \mathbb{R}$, which is invariant under dilations in the $\mathbb{R}_+$ variable.

As regards Theorem 1.2, we also show that if one fixes a point of a 5-manifold M, most generic rank 2 distributions satisfy that L is injective and A is 3-nondegenerate at that point. There is a normal form for such distributions: with respect to a suitable choice of local coordinates (x, y, z, p, q), an arbitrary generic 2-plane field can be written locally as

$$\mathcal{D} = \mathrm{span}\{\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z\} \tag{1.2}$$

for a scalar function F such that F_{qq} is nonvanishing (see [BH]). The coordinates can be taken so that the chosen point is the origin. For $\mathcal{D}$ in the form (1.2), in [N1], [N2], Nurowski gives a formula for a representative metric g_F of the conformal class such that the components of g_F and g_F^{-1} are polynomials in F, the derivatives of F of orders ≤ 4, and F_{qq}^{-1}, with coefficients which are universal functions of the local coordinates.

From this it is evident that the components of the Weyl and Cotton tensors of g_F can be expressed by similar formulae involving derivatives of F of orders ≤ 6, resp. ≤ 7.

Proposition 1.3. *Let $\mathcal{D}$ have the form* (1.2) *with F_{qq} nonvanishing. Each of the sets defined by the conditions* $\operatorname{rank}(L) < 6$ *at the origin, or A is 3-degenerate at the origin, is contained in a proper algebraic subvariety in the space of 7-jets of F at the origin.*

The union of these two sets is therefore contained in a proper Zariski closed set in the space of 7-jets at the origin, so its complement is dense. The Taylor expansion of F beyond order 7 can be chosen arbitrarily without affecting these conditions. By Theorem 1.2, if at any point of M the 7-jet of an F representing $\mathcal{D}$ lies in this complement, then $\operatorname{Hol}(\widetilde{g}) = G_2$. In fact, it is sufficient that the conditions be violated at different points.

The ambient metrics arising from distributions satisfying the hypotheses in Theorem 1.2 thus form an infinite-dimensional family of metrics whose holonomy is equal to G_2. Unfortunately, these metrics are not complete. They are "global" with respect to M, but arise as power series in the variable ρ in the last $\mathbb{R}$ factor whose radius of convergence may be small.

As explained in Chapter 4 of [FG2], the restriction of an ambient metric to $\{\rho > 0\}$ or $\{\rho < 0\}$ is a cone metric over a base which is called a Poincaré metric g_+. (This is another reason they are not complete.) The G_2 holonomy condition on $\widetilde{g}$ can be reintepreted in terms of g_+. Depending on the sign chosen for ρ, g_+ either has signature $(2,4)$ and is nearly Kähler of constant type 1, or has signature $(3,3)$ and is nearly para-Kähler of constant type 1. In particular, this gives new infinite-dimensional families of such metrics. This and other consequences of parallel tractor extension in the Poincaré metric setting will be the subject of a forthcoming paper by the second author.

The conditions in Theorem 1.2 are far from necessary for $\operatorname{Hol}(\widetilde{g}) = G_2$. Our goal was to find simple conditions which could be verified at a point and which give a dense set of 2-plane fields for which $\operatorname{Hol}(\widetilde{g}) = G_2$. Nurowski's whole family of examples lies in our complementary set even though almost all of them satisfy $\operatorname{Hol}(\widetilde{g}) = G_2$. In fact, any $\mathcal{D}$ in Nurowski's full 8-parameter family has the property that the tensor A is 2-degenerate at every point: at each point there exists $0 \neq X \in \mathcal{D}$ such that $A(Y_1, Y_2, X, X) = 0$ for all Y_1, $Y_2 \in \mathcal{D}$.

Theorem 1.2 is proved by quoting Theorem 1.1 and then using arguments developed by Leistner-Nurowski to rule out the possibility that the holonomy is strictly contained in G_2. In order to prove Theorem 1.1, we establish a result in a much more general setting which we think is of independent interest. We explain this next.

The analog in conformal geometry of the Levi-Civita connection in Riemannian geometry is the tractor connection. On a conformal manifold (M, c) of signature (p,q), $p+q=n$, there is a canonical rank $n+2$ vector bundle $\mathcal{T}$, the standard tractor bundle. It carries a metric of signature $(p+1, q+1)$ and a canonical connection,

the normal tractor connection, with respect to which the tractor metric is parallel. The holonomy of the tractor connection is therefore a subgroup of $O(p+1, q+1)$ and is referred to as the conformal holonomy of (M, c). Just as interesting classes of pseudo-Riemannian metrics can be described by holonomy reductions, interesting classes of conformal structures can be described by conformal holonomy reductions. A result of Hammerl-Sagerschnig [HS] shows that Nurowski's conformal structures have precisely such a characterization: an oriented conformal structure of signature $(2, 3)$ arises from a generic 2-plane field $\mathcal{D}$ if and only if its conformal holonomy is contained in G_2.

Just as pseudo-Riemannian holonomy reductions are often characterized by the existence of a parallel tensor, conformal holonomy reductions are often characterized by the existence of a parallel tractor (by which we here mean a parallel section of $\otimes^r \mathcal{T}^*$ for some $r > 0$). Since G_2 is defined as the subgroup of $GL(7, \mathbb{R})$ preserving a 3-form compatible with a metric of signature $(3, 4)$, the Hammerl-Sagerschnig holonomy criterion can be reinterpreted (as they do) as the condition that (M, c) admit a parallel tractor 3-form (i.e. a section of $\Lambda^3 \mathcal{T}^*$) compatible with the tractor metric. Likewise, in order to show that $\mathrm{Hol}(\widetilde{g}) \subset G_2$, one needs to show that there is a parallel 3-form on the ambient space compatible with $\widetilde{g}$. Thus one is led to the problem of constructing a parallel 3-form on the ambient space given a parallel tractor 3-form.

There are several constructions of the standard tractor bundle and its metric and connection. These were originally defined by T. Y. Thomas in [T] in language that predates the definition of a vector bundle. The first modern treatment is [BEG]. The paper [ČG1] explains the relation between tractors and the ambient construction. Recall that the ambient metric $\widetilde{g}$ is defined on an open subset $\widetilde{\mathcal{G}} \subset \mathbb{R}_+ \times M \times \mathbb{R}$ containing $\mathbb{R}_+ \times M \times \{0\}$. The hypersurface $\mathcal{G} = \mathbb{R}_+ \times M \times \{0\} \subset \widetilde{\mathcal{G}}$ can be viewed as an $\mathbb{R}_+$-bundle over M. A tractor (section of $\mathcal{T}$) on M can be regarded as a section of the bundle $T\widetilde{\mathcal{G}}|_{\mathcal{G}}$ over $\mathcal{G}$ with a particular homogeneity with respect to the $\mathbb{R}_+$-dilations. The tractor metric and connection can be realized as the restriction to $\mathcal{G}$ of the ambient metric $\widetilde{g}$ and its Levi-Civita connection. That is, the restriction to $\mathcal{G}$ of a tensor field on $\widetilde{\mathcal{G}}$ homogeneous of the correct degree with respect to the $\mathbb{R}_+$ dilations defines a tractor field on M. The condition that the tractor field is parallel with respect to the tractor connection is precisely the condition that the restriction to $\mathcal{G}$ of the tensor field have zero covariant derivative with respect to the ambient connection when the differentiations are taken in directions tangent to $\mathcal{G}$. Thus the problem described above of constructing a parallel 3-form on the ambient space given a parallel tractor 3-form amounts to extending such a "tangentially parallel" ambient 3-form defined on the hypersurface $\mathcal{G}$ to a parallel 3-form on $\widetilde{\mathcal{G}}$.

We prove a "parallel tractor extension theorem" of this nature for general tractors irrespective of their rank, symmetry or algebraic type on conformal manifolds of any dimension and signature. For smooth odd dimensional conformal manifolds, the ambient metric is determined by the conformal structure to infinite order along $\mathcal{G}$. For

smooth even-dimensional conformal manifolds, it is only determined to order $n/2-1$. These indeterminacies in the ambient metric are reflected in the statement of the parallel tractor extension theorem.

Theorem 1.4. *Let (M,c) be a conformal manifold of dimension $n \geq 3$ and let $\widetilde{g}$ be an ambient metric for (M,c). Let $r \in \mathbb{N}$ and suppose $\chi \in \Gamma\left(\otimes^r \mathcal{T}^*\right)$ satisfies $\nabla\chi = 0$, where ∇ denotes the tractor covariant derivative.*

- *If n is odd, then χ has an ambient extension $\widetilde{\chi}$ satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^\infty)$.*
- *If n is even, then χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2-1})$.*

The proof shows that for n odd, if (M,c) and $\widetilde{g}$ are real analytic, then $\widetilde{\chi}$ may be taken to be real-analytic so that $\widetilde{\nabla}\widetilde{\chi} = 0$ in a neighborhood of $\mathcal{G}$.

Theorem 1.4 was proved for n odd in [Go3] for the case $r = 1$ using an argument based on "harmonic extension". The same argument also proves Theorem 1.4 for n even and $r = 1$. Results essentially containing Theorem 1.4 in case c contains an Einstein metric are proved in [Leit1], [Leis]; see the discussion in §4 below.

Our proof of Theorem 1.4 goes by first extending χ by parallel translation along the lines $\rho \mapsto (z,\rho)$ for each $z \in \mathcal{G}$. It must be shown that this parallel translation preserves the vanishing covariant derivatives in the tangential directions. This involves commutation arguments which use consequences to high order of the homogeneity and Ricci-flatness of the ambient metric.

In [Br2], Bryant showed that for generic 3-plane fields on 6-manifolds there is a construction analogous to Nurowski's construction: a generic 3-plane field induces a conformal structure of signature $(3,3)$ on the same 6-manifold. It is tempting to speculate about the possibility of constructing signature $(4,4)$ metrics of holonomy $\mathrm{Spin}(3,4)$ as ambient metrics of such conformal manifolds. This would require finding extensions parallel to infinite order for n even in Theorem 1.4.

Partly with such considerations in mind, we investigate here some beginning cases of what can happen in even dimensions concerning parallel extension beyond order $n/2-1$. There are several issues. One complication is that in order to construct ambient metrics which are Ricci-flat to higher order, it is in general necessary to include log terms in the expansion of $\widetilde{g}$. To avoid this complication we mostly restrict attention here to the case of vanishing obstruction tensor, for which log terms do not enter. In this case, the proof of Theorem 1.4 shows that $n/2$ is the critical order: if a parallel tractor has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$, it has an extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^\infty)$. Another complication in even dimensions is that higher-order ambient metrics are no longer determined by the conformal structure alone: there is an ambiguity at order $n/2$ in the ambient metric. So whether or not a parallel tractor has a parallel ambient extension may depend on which ambient metric one chooses. In §4 we investigate this for three classes of conformal structures admitting parallel tractors: conformal classes containing an Einstein metric, Poincaré-Einstein conformal classes, and Fefferman conformal structures associated

to nondegenerate hypersurfaces in $\mathbb{C}^n$. We find that for conformal classes containing an Einstein metric, there is always a unique choice of ambiguity for which there is a parallel ambient extension (Proposition 4.4). For Poincaré-Einstein conformal classes we give necessary and sufficient conditions on the Poincaré-Einstein metric for there to exist an ambient metric for which the parallel tractor has a parallel ambient extension (Proposition 4.5), and using a recent result of Juhl ([J]) we give a formula for the distinguished ambient metric and the parallel extension when they exist (Proposition 4.6). In particular, Proposition 4.5 gives examples of parallel tractors which have no ambient extension parallel to order $n/2$ for any choice of ambiguity. For Fefferman conformal structures we show that the parallel tractor 2-form has a parallel extension for infinitely many choices of ambiguity in the ambient metric (Proposition 4.9).

The organization of the paper is as follows. In §2 we review background material concerning ambient metrics and tractors. In §3 we prove Theorem 1.4. As a consequence of Theorem 1.4 we derive a sequence of integrability conditions to higher and higher order which must be satisfied by any parallel tractor. §4 studies parallel extension beyond order $n/2-1$ in even dimensions as described above. We formulate a general condition on a tractor which we call "determining" which guarantees in the case of vanishing obstruction tensor that there is at most one choice of ambient metric with respect to which the parallel tractor has an ambient extension parallel to order $n/2$. In §5 we discuss background concerning generic 2-plane fields, Nurowski's conformal structures, and the work of Leistner-Nurowski and Hammerl-Sagerschnig. We show how Cartan's tensor A can be realized as a piece of the Weyl curvature of Nurowski's conformal structure and we prove Theorems 1.1 and 1.2 and Proposition 1.3. §6 is an appendix in which we give our conventions concerning G_2, collect facts about Cartan's connection and curvature in the form given by Nurowski [N1], and prove two facts about the curvature which are used in §5.

Some of the results in this paper are contained in the Ph.D. thesis of the second author ([W]). This thesis includes more details and further results in certain directions.

2. Ambient Metrics and Tractors

In this section we review background material concerning ambient metrics and tractors. The main reference for the material on ambient metrics is [FG2], and references for the approach taken here for tractors are [ČG1] and [BG].

Let (M, c) be a conformal manifold of dimension $n \geq 3$ and signature (p, q), $p+q = n$. This means that c is an equivalence class of metrics under the relation $g \sim \Omega^2 g$ for $0 < \Omega \in C^\infty(M)$. The metric bundle of (M, c) is by definition $\mathcal{G} := \{(x, g_x) : x \in M, g \in c\} \subset S^2T^*M$. Let $\pi : \mathcal{G} \to M$ denote the projection. There is an action of $\mathbb{R}_+$ on $\mathcal{G}$ defined by $\delta_s(x, g_x) = (x, s^2 g_x)$ for $s \in \mathbb{R}_+$. Let $T = \frac{d}{ds}\delta_s|_{s=1}$ be the infinitesimal generator of the $\mathbb{R}_+$ action. There is a tautological symmetric 2-tensor $\mathbf{g}_0$ on $\mathcal{G}$ defined for X, $Y \in T_{(x,g_x)}\mathcal{G}$ by $\mathbf{g}_0(X, Y) = g_x(\pi_*X, \pi_*Y)$.

Consider the space $\mathcal{G} \times \mathbb{R}$. The variable in the $\mathbb{R}$ factor is usually denoted ρ. The dilations δ_s extend to $\mathcal{G} \times \mathbb{R}$ acting in the $\mathcal{G}$ factor, and we denote also by T the infinitesimal generator on $\mathcal{G} \times \mathbb{R}$. The map $\iota : \mathcal{G} \to \mathcal{G} \times \mathbb{R}$ defined for $z \in \mathcal{G}$ by $\iota(z) = (z, 0)$ imbeds $\mathcal{G}$ as a hypersurface in $\mathcal{G} \times \mathbb{R}$.

A smooth metric $\widetilde{g}$ of signature $(p+1, q+1)$ on a dilation-invariant open neighborhood $\widetilde{\mathcal{G}}$ of $\mathcal{G} \times \{0\}$ in $\mathcal{G} \times \mathbb{R}$ is said to be a pre-ambient metric for (M, c) if it satisfies the following two conditions:

(1) $\delta_s^* \widetilde{g} = s^2 \widetilde{g} \quad$ for $s \in \mathbb{R}_+$;
(2) $\iota^* \widetilde{g} = \mathbf{g}_0$.

A pre-ambient metric $\widetilde{g}$ is said to be straight if for each $p \in \widetilde{\mathcal{G}}$ the parametrized curve $s \mapsto \delta_s p$ is a geodesic for $\widetilde{g}$. This is equivalent to the condition that $\widetilde{\nabla} T = Id$ where Id denotes the identity endomorphism and $\widetilde{\nabla}$ the Levi-Civita connection of $\widetilde{g}$; see Propositions 2.4 and 3.4 of [FG2].

If n is odd, an ambient metric for (M, c) is a straight pre-ambient metric for (M, c) such that $\mathrm{Ric}(\widetilde{g})$ vanishes to infinite order on $\mathcal{G} \times \{0\}$. (The straightness condition is automatic to infinite order, but it is convenient to include it in the definition.) There exists an ambient metric for (M, c) and it is unique to infinite order up to pullback by a diffeomorphism defined on a dilation-invariant neighborhood of $\mathcal{G} \times \mathbb{R}$ which commutes with dilations and which restricts to the identity on $\mathcal{G} \times \{0\}$. If M is a real-analytic manifold and there is a real-analytic metric in the conformal class, then there exists a real-analytic ambient metric for (M, c) satisfying $\mathrm{Ric}(\widetilde{g}) = 0$ on some dilation-invariant $\widetilde{\mathcal{G}}$ as above.

In order to formulate the definition of ambient metrics for n even, if S_{IJ} is a symmetric 2-tensor field on an open neighborhood of $\mathcal{G} \times \{0\}$ in $\mathcal{G} \times \mathbb{R}$ and $m \geq 0$, we write $S_{IJ} = O^+_{IJ}(\rho^m)$ if $S_{IJ} = O(\rho^m)$ and for each point $z \in \mathcal{G}$, the symmetric 2-tensor $(\iota^*(\rho^{-m} S))(z)$ is of the form $\pi^* s$ for some symmetric 2-tensor s at $x = \pi(z) \in M$ satisfying $\mathrm{tr}_{g_x} s = 0$. The symmetric 2-tensor s is allowed to depend on z, not just on x. If n is even, an ambient metric for (M, c) is a straight pre-ambient metric such that $\mathrm{Ric}(\widetilde{g}) = O^+_{IJ}(\rho^{n/2-1})$. There exists an ambient metric for (M, c) and it is unique up to addition of a term which is $O^+_{IJ}(\rho^{n/2})$ and up to pullback by a diffeomorphism defined on a dilation-invariant neighborhood of $\mathcal{G} \times \mathbb{R}$ which commutes with dilations and which restricts to the identity on $\mathcal{G} \times \{0\}$.

The diffeomorphism invariance of ambient metrics can be broken by putting them into a normal form with respect to a choice of metric g in the conformal class. Observe first that the choice of $g \in c$ determines a trivialization of the bundle $\mathcal{G} \to M$ by identifying $(t, x) \in \mathbb{R}_+ \times M$ with $(x, t^2 g_x) \in \mathcal{G}$. Under this identification the tautological tensor $\mathbf{g}_0$ takes the form $\mathbf{g}_0 = t^2 g$, where we omit writing π^* for the pullback of a tensor on M to $\mathcal{G}$, and we have $T = t\partial_t$. There is an induced identification $\mathcal{G} \times \mathbb{R} \cong \mathbb{R}_+ \times M \times \mathbb{R}$. A pre-ambient metric $\widetilde{g}$ is said to be in normal form with respect to $g \in c$ if it satisfies the following three conditions:

(1) Its domain of definition $\widetilde{\mathcal{G}}$ has the property that for each $z \in \mathcal{G}$, the set of $\rho \in \mathbb{R}$ such that $(z,\rho) \in \widetilde{\mathcal{G}}$ is an open interval I_z containing 0;
(2) For each $z \in \mathcal{G}$, the parametrized curve $I_z \ni \rho \mapsto (z,\rho)$ is a geodesic for $\widetilde{g}$;
(3) Under the identification $\mathcal{G} \times \mathbb{R} \cong \mathbb{R}_+ \times M \times \mathbb{R}$ induced by g, at each point $(t,x,0) \in \mathcal{G} \times \{0\}$, $\widetilde{g}$ takes the form $\widetilde{g} = t^2 g + 2tdtd\rho$.

A straight pre-ambient metric is in normal form with respect to g if and only if it has the form

$$\widetilde{g} = 2tdtd\rho + 2\rho\, dt^2 + t^2 g_\rho \tag{2.1}$$

relative to the identification $\mathcal{G} \times \mathbb{R} \cong \mathbb{R}_+ \times M \times \mathbb{R}$ induced by g, where g_ρ is a smooth family of metrics on M parametrized by ρ satisfying $g_0 = g$. Any pre-ambient metric can be put into normal form with respect to a choice of $g \in c$ by a unique diffeomorphism which commutes with the dilations and restricts to the identity on $\mathcal{G} \times \{0\}$. For n odd, the existence and uniqueness assertion for ambient metrics in normal form states that given a metric g on M, there exists an ambient metric $\widetilde{g}$ for $(M,[g])$ in normal form with respect to g, and g_ρ in (2.1) is uniquely determined to infinite order at $\rho = 0$. For n even, the corresponding assertion is that g_ρ is uniquely determined mod $O(\rho^{n/2})$ and also $\operatorname{tr}_g \left(\partial_\rho^{n/2} g_\rho|_{\rho=0}\right)$ is determined. In all dimensions $n \geq 3$ one has

$$g_\rho = g + 2P\rho + O(\rho^2) \tag{2.2}$$

where P denotes the Schouten tensor of g, defined by

$$(n-2)P_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}.$$

For n even a conformally invariant tensor, the ambient obstruction tensor, obstructs the existence of smooth solutions to $\operatorname{Ric}(\widetilde{g}) = O(\rho^{n/2})$. However if the obstruction tensor vanishes then there are smooth solutions to higher order. (In general there are higher-order solutions with expansions involving log terms; see Theorem 3.10 of [FG2].) If (M,c) is a conformal manifold of even dimension $n \geq 4$, by an infinite-order ambient metric we will mean a smooth straight pre-ambient metric for which $\operatorname{Ric}(\widetilde{g})$ vanishes to infinite order at $\rho = 0$. If (M,c) admits an infinite-order ambient metric, then it has vanishing obstruction tensor. The Taylor expansion of an infinite-order ambient metric in normal form is no longer determined solely by the initial metric g. It follows from Theorem 3.10 of [FG2] that there is a natural pseudo-Riemannian invariant 1-form $D_i(g)$ depending on a metric g, so that if g has vanishing obstruction tensor and κ is a smooth symmetric 2-tensor on M which is trace-free with respect to g and satisfies $\kappa_{ij},{}^j = D_i(g)$ where the divergence is with respect to the Levi-Civita connection of g, then there is an infinite-order ambient metric in normal form with respect to g such that $\operatorname{tf}\left(\partial_\rho^{n/2} g_\rho|_{\rho=0}\right) = \kappa$. Here tf denotes the trace-free part with respect to g. Moreover, these conditions uniquely determine g_ρ to infinite order at

$\rho = 0$ and all infinite-order ambient metrics in normal form relative to g arise from such a κ. We will call κ the ambiguity in the infinite-order ambient metric.

We will use capital Latin indices to label objects on $\mathcal{G} \times \mathbb{R}$. Upon choosing a metric $g \in c$ we have the splitting $\mathcal{G} \times \mathbb{R} \cong \mathbb{R}_+ \times M \times \mathbb{R}$. We will use a 0 index for the $\mathbb{R}_+$ factor, lower case Latin indices for the M factor, and an ∞ index for the $\mathbb{R}$ factor.

For the purposes of this paper it will be convenient to define the tractor bundle and connection in ambient terms. Such a formulation was given in [ČG1], [BG] where further discussion and details may be found.

Let (M, c) be a conformal manifold with metric bundle $\mathcal{G} \xrightarrow{\pi} M$. For $x \in M$, write $\mathcal{G}_x = \pi^{-1}(\{x\})$ for the fiber of $\mathcal{G}$ over x. Recall that the bundle $\mathcal{D}(w)$ of conformal densities of weight $w \in \mathbb{C}$ has fiber $\mathcal{D}_x(w) = \{f : \mathcal{G}_x \to \mathbb{C} : (\delta_s)^* f = s^w f,\ s > 0\}$, so that sections of $\mathcal{D}(w)$ on M are functions on $\mathcal{G}$ homogeneous of degree w. A metric g in the conformal class is a section of $\mathcal{G}$, so if f is a section of $\mathcal{D}(w)$, then $f \circ g$ is a function on M. Under conformal change $\widehat{g} = \Omega^2 g$, we have $f \circ \widehat{g} = \Omega^w f \circ g$.

The standard tractor bundle of (M, c) and its metric and connection can be similarly defined in terms of homogeneous vector fields on $\mathcal{G}_x$. Identify $\mathcal{G}$ with the subset $\mathcal{G} \times \{0\} \subset \mathcal{G} \times \mathbb{R}$ via the map ι. Let $\widetilde{g}$ be an ambient metric for (M, c) defined on a dilation-invariant open neighborhood $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ in $\mathcal{G} \times \mathbb{R}$. Consider the rank $n+2$ vector bundle $\mathcal{T} \to M$ with fiber

$$\mathcal{T}_x = \left\{ U \in \Gamma(T\widetilde{\mathcal{G}}\big|_{\mathcal{G}_x}) : (\delta_s)_* U = sU,\ s > 0 \right\}. \tag{2.3}$$

So a section of $\mathcal{T}$ on M is the same as a section U of $T\widetilde{\mathcal{G}}\big|_{\mathcal{G}}$ on $\mathcal{G}$ satisfying $(\delta_s)_* U = sU$, or equivalently $(\delta_s)^* U = s^{-1} U$. If $U,\ W \in \mathcal{T}_x$, then $\widetilde{g}(U, W)$ is homogeneous of degree 0 on $\mathcal{G}_x$, i.e. $\widetilde{g}(U, W) \in \mathbb{R}$. This therefore defines a metric h of signature $(p+1, q+1)$ on $\mathcal{T}$. Since T is homogeneous of degree 0 with respect to the δ_s, it defines a section of $\mathcal{T}(1) := \mathcal{T} \otimes \mathcal{D}(1)$. But the set of U which at each point of $\mathcal{G}_x$ is a multiple of T constitutes a subbundle of $\mathcal{T}$ which we denote $\operatorname{span}\{T\}$. Its orthogonal complement $\operatorname{span}\{T\}^\perp$ is the set of U which at each point of $\mathcal{G}_x$ is tangent to $\mathcal{G}$. This gives the filtration

$$0 \subset \operatorname{span}\{T\} \subset \operatorname{span}\{T\}^\perp \subset \mathcal{T}. \tag{2.4}$$

In order to realize the tractor connection, observe that $\pi_* : T\mathcal{G} \to TM$ induces a realization of the tangent bundle TM as

$$T_x M = \left\{ V \in \Gamma(T\mathcal{G}\big|_{\mathcal{G}_x}) : (\delta_s)_* V = V,\ s > 0 \right\} \Big/ \operatorname{span}\{T\},$$

where now $\operatorname{span}\{T\}$ really means the constant multiples of T. If $v \in T_x M$, choose $V \in \Gamma(T\mathcal{G}\big|_{\mathcal{G}_x})$ representing v. If U is a section of $\mathcal{T}$ near x, define the tractor connection ∇ by $\nabla_v U = \widetilde{\nabla}_V U$. Observe first that the right-hand side makes sense since $U \in \Gamma(T\widetilde{\mathcal{G}}\big|_{\mathcal{G}})$ and V is tangent to $\mathcal{G}$. To see that the right-hand side is independent of

the choice of V representing v it suffices to show that $\widetilde{\nabla}_T U = 0$. Now

$$\widetilde{\nabla}_T U = \widetilde{\nabla}_U T + [T, U] = \widetilde{\nabla}_U T + \mathcal{L}_T U$$

where $\mathcal{L}$ denotes the Lie derivative. But $\mathcal{L}_T U = -U$ by the homogeneity of U and $\widetilde{\nabla}_U T = U$ since $\widetilde{g}$ is straight. Thus $\widetilde{\nabla}_T U = 0$ as desired. Finally, $\widetilde{\nabla}_V U$ has the same homogeneity as U since V and $\widetilde{\nabla}$ are invariant under the δ_s: V is invariant by hypothesis and $\widetilde{\nabla}$ is invariant since δ_s is a homothety of $\widetilde{g}$. Thus $\widetilde{\nabla}_V U$ is a well-defined section of $\mathcal{T}$ and it is easily checked that this defines a connection. The tractor metric is parallel with respect to ∇ since $\widetilde{\nabla}\widetilde{g} = 0$. But the filtration (2.4) is not ∇-parallel.

Note that this realization of the tractor bundle and connection depends on the choice of ambient metric $\widetilde{g}$. If $\widetilde{g}$ is changed by a diffeomorphism, one obtains different but equivalent realizations.

It is shown in [ČG1] that this formulation of the standard tractor bundle and connection agrees with other definitions by using a functorial characterization of tractor bundles. For our purposes it will be useful to see this in terms of the splitting induced by a choice of g. As we did with conformal densities, we associate to $U \in \Gamma(\mathcal{T})$ the map $U \circ g$ defined on M, for which $(U \circ g)(x) \in T_{(x,g_x)}\widetilde{\mathcal{G}}$. We decompose $T_{(x,g_x)}\widetilde{\mathcal{G}}$ via the splitting in which $\widetilde{g}$ is in normal form with respect to g. That is, after composing with a diffeomorphism if necessary, we assume that $\widetilde{g}$ is in normal form with respect to g. The splitting $\mathcal{G} \times \mathbb{R} \cong \mathbb{R}_+ \times M \times \mathbb{R}$ induces a splitting $T(\mathcal{G} \times \mathbb{R}) \cong T\mathbb{R}_+ \oplus TM \oplus T\mathbb{R}$. Via the trivializations of $T\mathbb{R}_+$ and $T\mathbb{R}$ induced by ∂_t and ∂_ρ, resp., $U \circ g$ is expressed in the form (U^0, U^i, U^∞), where U^0 and U^∞ are functions on M and U^i is a vector field on M. In terms of coordinates (t, x, ρ) with respect to which $\widetilde{g}$ is in normal form, we are simply writing $U \circ g = U^0\partial_t + U^i\partial_{x^i} + U^\infty\partial_\rho$. In this way, relative to the choice of g we represent a section $U \in \Gamma(\mathcal{T})$ by the triple (U^0, U^i, U^∞). Recalling that $t = 1$ on points of $\mathcal{G}$ of the form (x, g_x), it is evident from condition (3) in the definition of normal form that the tractor metric is given by $h(U, U) = 2U^0U^\infty + g_{ij}U^iU^j$.

Suppose now we make a conformal change to $\widehat{g} = e^{2\Upsilon}g$. The ambient metric $\widetilde{g}$ can be put into normal form relative to $\widehat{g}$ by pulling back by a homogeneous diffeomorphism. Arguing as in the proof of Proposition 6.5 of [FG2], one can identify the Jacobian on $\mathcal{G}$ of the diffeomorphism and thus calculate the relation between the representation $(\widehat{U}^0, \widehat{U}^i, \widehat{U}^\infty)$ of U with respect to $\widehat{g}$ and that with respect to g. The result is:

$$\begin{pmatrix} \widehat{U}^0 \\ \widehat{U}^i \\ \widehat{U}^\infty \end{pmatrix} = \begin{pmatrix} e^{-\Upsilon} & 0 & 0 \\ 0 & e^{-\Upsilon} & 0 \\ 0 & 0 & e^{\Upsilon} \end{pmatrix} \begin{pmatrix} 1 & -\Upsilon_j & -\frac{1}{2}\Upsilon_k\Upsilon^k \\ 0 & \delta^i{}_j & \Upsilon^i \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U^0 \\ U^j \\ U^\infty \end{pmatrix}.$$

This is the identification used in the construction of the standard tractor bundle in [BEG], so shows the agreement of these constructions.

The tractor connection can also be expressed in terms of the splitting. It is straightforward to calculate the Christoffel symbols of a metric of the form (2.1). One obtains:

$$
\begin{aligned}
\widetilde{\Gamma}^0_{IJ} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}tg'_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\widetilde{\Gamma}^k_{IJ} &= \begin{pmatrix} 0 & t^{-1}\delta_j{}^k & 0 \\ t^{-1}\delta_i{}^k & \Gamma^k_{ij} & \frac{1}{2}g^{kl}g'_{il} \\ 0 & \frac{1}{2}g^{kl}g'_{jl} & 0 \end{pmatrix} \\
\widetilde{\Gamma}^\infty_{IJ} &= \begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & -g_{ij}+\rho g'_{ij} & 0 \\ t^{-1} & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{2.5}
$$

(See (3.16) of [FG2].) Here all g_{ij} and g^{ij} refer to g_ρ, $'$ denotes ∂_ρ, Γ^k_{ij} denotes the Christoffel symbol of the metric g_ρ with ρ fixed, and the blocks correspond to the splittings $I \leftrightarrow (0, i, \infty)$, $J \leftrightarrow (0, j, \infty)$. In order to represent $\nabla_v U = \widetilde{\nabla}_V U$ in terms of the splitting with respect to g, we evaluate (2.5) at $\rho = 0$, $t = 1$ and consider only the $I = i$ components. Recalling (2.2), this gives for the tractor Christoffel symbols:

$$
\begin{aligned}
\widetilde{\Gamma}^0_{iJ} &= \begin{pmatrix} 0 & -P_{ij} & 0 \end{pmatrix} \\
\widetilde{\Gamma}^k_{iJ} &= \begin{pmatrix} \delta_i{}^k & \Gamma^k_{ij} & P_i{}^k \end{pmatrix} \\
\widetilde{\Gamma}^\infty_{iJ} &= \begin{pmatrix} 0 & -g_{ij} & 0 \end{pmatrix}.
\end{aligned}
\tag{2.6}
$$

The tractor covariant derivative is given by $\nabla_i U^K = \partial_i U^K + \widetilde{\Gamma}^K_{iJ} U^J$, or equivalently

$$
\nabla_i \begin{pmatrix} U^0 \\ U^k \\ U^\infty \end{pmatrix} = \begin{pmatrix} \nabla_i U^0 - P_{ij}U^j \\ \nabla_i U^k + \delta_i{}^k U^0 + P_i{}^k U^\infty \\ \nabla_i U^\infty - U_i \end{pmatrix}.
\tag{2.7}
$$

On the right-hand side, $\nabla_i U^0$ and $\nabla_i U^\infty$ denote the exterior derivative on functions and $\nabla_i U^k$ the Levi-Civita connection of g on vector fields on M. Equation (2.7) is taken as the definition of the tractor connection in [BEG], so the ambient construction produces the usual (normal) tractor connection.

3. Parallel Extension

In this section we prove Theorem 1.4. We will be dealing primarily with cotractors. The realization dual to (2.3) is

$$
\mathcal{T}^*_x = \left\{ \eta \in \Gamma(T^*\widetilde{\mathcal{G}}\big|_{\mathcal{G}_x}) : (\delta_s)^*\eta = s\eta,\ s > 0 \right\}
$$

so that $\eta(U)$ is homogeneous of degree 0 on $\mathcal{G}_x$. Hence if $r \in \mathbb{N}$, we realize sections of $\otimes^r \mathcal{T}^*$ as $\chi \in \Gamma(\otimes^r T^*\widetilde{\mathcal{G}}\big|_{\mathcal{G}})$ satisfying $(\delta_s)^*\chi = s^r\chi$. Any such χ satisfies $\widetilde{\nabla}_T \chi = 0$ and the cotractor connection is realized by $\nabla_v \chi = \widetilde{\nabla}_V \chi$ by analogy with the discussion

for tractors in §2. We use the splitting for $\mathcal{T}^*$ dual to the one above: $\chi \in \mathcal{T}^*$ is represented with respect to $g \in c$ by $\chi = (\chi_0, \chi_i, \chi_\infty)$ if $\chi \circ g = \chi_0 dt + \chi_i dx^i + \chi_\infty d\rho$.

If $\chi \in \Gamma(\otimes^r \mathcal{T}^*)$, we say that $\widetilde{\chi} \in \Gamma(\otimes^r T^*\widetilde{\mathcal{G}})$ is an ambient extension of χ if $\delta_s^* \widetilde{\chi} = s^r \widetilde{\chi}$ and $\widetilde{\chi}|_{\mathcal{G}} = \chi$. Clearly if $\widetilde{\nabla}\widetilde{\chi} = 0$ then restricting to $\rho = 0$ and to differentiations tangent to $\mathcal{G}$, it follows that $\nabla\chi = 0$. Theorem 1.4 asserts that any parallel tractor admits an ambient extension which is as parallel as one can expect given the indeterminacy in the ambient metric.

Before beginning the proof of Theorem 1.4, observe that uniqueness of the asserted extension is easy: if n is odd then $\widetilde{\chi}$ is unique to infinite order and if n is even then $\widetilde{\chi} \mod O(\rho^{n/2})$ is uniquely determined. This follows just from the fact that $\widetilde{\nabla}_\infty \widetilde{\chi}$ vanishes to the stated order by successively differentiating with respect to ρ at $\rho = 0$ the vanishing condition applied to the difference. (See the proof of Proposition 3.2 below.) Note also that since $\widetilde{\nabla}\widetilde{\chi}$ depends on first derivatives of $\widetilde{g}$, if n is even then the indeterminacy of $\widetilde{g}$ at order $n/2$ enters into $\widetilde{\nabla}\widetilde{\chi}$ at order $n/2$. This is the subject of §4.

The proof of Theorem 1.4 uses properties of the covariant derivatives of the curvature tensor of an ambient metric. We denote the curvature tensor of a pre-ambient metric by $\widetilde{R}$ and covariant derivatives with respect to its connection by indices preceded by a comma. Proposition 6.1 of [FG2] asserts that the curvature tensor of any straight pre-ambient metric satisfies for $r \geq 0$

$$
\begin{aligned}
T^L \widetilde{R}_{IJKL,M_1\cdots M_r} &= -\sum_{s=1}^{r} \widetilde{R}_{IJKM_s,M_1\cdots\widehat{M_s}\cdots M_r} \\
T^P \widetilde{R}_{IJKL,PM_1\cdots M_r} &= -2\widetilde{R}_{IJKL,M_1\cdots M_r} - \sum_{s=1}^{r} \widetilde{R}_{IJKL,M_sM_1\cdots\widehat{M_s}\cdots M_r}.
\end{aligned} \tag{3.1}
$$

The empty sum on the right-hand sides is interpreted as 0 in case $r = 0$. Suppose next that $\widetilde{g}$ is an ambient metric in normal form. The indices can be specialized according to the normal form splitting $I \leftrightarrow (0, i, \infty)$.

Lemma 3.1. *If $\widetilde{g}$ is an ambient metric in normal form, then at $\rho = 0$, $t = 1$ we have*

$$(2k+1)\widetilde{R}_{IJA\infty,\underbrace{\infty\cdots\infty}_{k-1}} = \widetilde{R}_{IJA}{}^{p}{}_{,p\underbrace{\infty\cdots\infty}_{k-1}}.$$

This holds for all $k \geq 1$ if n is odd and for IJA and k satisfying $\|IJA\| + 2k \leq n+1$ if n is even, where $\|0\| = 0$, $\|i\| = 1$ for $1 \leq i \leq n$, $\|\infty\| = 2$, and $\|IJA\| = \|I\| + \|J\| + \|A\|$.

Proof. First assume that n is odd. The second Bianchi identity and the fact that $\widetilde{g}$ is Ricci flat to infinite order imply at $\rho = 0$

$$\widetilde{g}^{PQ}\widetilde{R}_{IJAP,Q\underbrace{\infty\cdots\infty}_{k-1}} = 0$$

for all k. Write out the trace using the normalization of $\widetilde{g}$ at $\rho = 0$, $t = 1$ to obtain

$$\widetilde{R}_{IJA0,\infty\underbrace{\infty\cdots\infty}_{k-1}} + \widetilde{R}_{IJA\infty,0\underbrace{\infty\cdots\infty}_{k-1}} + g^{pq}\widetilde{R}_{IJAp,q\underbrace{\infty\cdots\infty}_{k-1}} = 0.$$

Now (3.1) shows that

$$\widetilde{R}_{IJA0,\underbrace{\infty\cdots\infty}_{k}} = -k\widetilde{R}_{IJA\infty,\underbrace{\infty\cdots\infty}_{k-1}}, \qquad \widetilde{R}_{IJA\infty,0\underbrace{\infty\cdots\infty}_{k-1}} = -(k+1)\widetilde{R}_{IJA\infty,\underbrace{\infty\cdots\infty}_{k-1}}$$

so the result follows.

If n is even, the hypothesis $\|IJA\| + 2k \le n+1$ guarantees that the relevant Ricci curvature derivative component vanishes so that the same argument applies. See Proposition 6.4 of [FG2]. $\square$

Proof of Theorem 1.4. Choose a metric g in the conformal class. Put $\widetilde{g}$ into normal form relative to g. The hypothesis $\nabla\chi = 0$ is equivalent to the statement that $\widetilde{\nabla}_A\widetilde{\chi}|_{\rho=0} = 0$ for $A = 0, a$ for any extension. Define $\widetilde{\chi}$ by parallel translation of χ along the lines $\rho \mapsto (t, x, \rho)$. Parallel translation commutes with the dilations so $\widetilde{\chi}$ has the right homogeneity. It suffices to show that $\widetilde{\nabla}\widetilde{\chi}$ vanishes to the stated order. The point is that the system $\widetilde{\nabla}\widetilde{\chi} = 0$ is overdetermined, so it must be shown that parallel translation in the ρ-direction preserves vanishing of the tangential covariant derivatives. This requires study of the commutation of ambient covariant derivatives to high order. Note that if (M, g) is real-analytic, then any parallel tractor χ is real-analytic, so if $\widetilde{g}$ is real-analytic, then $\widetilde{\chi}$ is real-analytic as well.

We show first that $\widetilde{\chi}_{\mathcal{I},\underbrace{\infty\cdots\infty}_{k}} = 0$ on $\widetilde{\mathcal{G}}$ for $k \ge 1$, where $\mathcal{I} = I_1 \cdots I_r$ is any list of r indices. This follows by induction on k. The case $k = 1$ is clear since $\widetilde{\chi}$ was defined to be parallel in the ρ-direction. For $k > 1$ write

$$\widetilde{\chi}_{\mathcal{I},\underbrace{\infty\cdots\infty}_{k}} = \partial_\infty\widetilde{\chi}_{\mathcal{I},\underbrace{\infty\cdots\infty}_{k-1}} - \sum_{s=1}^{r}\widetilde{\Gamma}^J_{I_s\infty}\widetilde{\chi}_{I_1\cdots I_{s-1}JI_{s+1}\cdots I_r,\underbrace{\infty\cdots\infty}_{k-1}} - \sum_{l=1}^{k-1}\widetilde{\Gamma}^A_{\infty\infty}\widetilde{\chi}_{\mathcal{I},\underbrace{\infty\cdots\infty}_{l-1}A\underbrace{\infty\cdots\infty}_{k-l-1}}.$$

The first two terms vanish by the induction hypothesis and the last because $\widetilde{\Gamma}^A_{\infty\infty} = 0$ from (2.5).

We argue next that if $\widetilde{\nabla}^l\widetilde{\chi}|_{\rho=0} = 0$ for $1 \le l \le k-1$, then $\widetilde{\nabla}^{k+1}\widetilde{\chi}|_{\rho=0}$ is symmetric in the last k of the $k+1$ differentiation indices. (We think of the differentiation indices as listed after the indices of $\widetilde{\chi}$, separated by a comma.) This follows by commuting two adjacent indices among the last k in $\widetilde{\nabla}^{k+1}\widetilde{\chi}|_{\rho=0}$. This commutation of derivatives can be written by the differentiated Ricci identity as a sum of terms of the form derivatives of curvature contracted into derivatives of $\widetilde{\chi}$, in which the total number of derivatives of $\widetilde{\chi}$ is at least 1 and drops by at least 2. Thus the result follows.

Now we proceed with the main induction: we prove by induction on k the statement $\widetilde{\nabla}^k\widetilde{\chi}|_{\rho=0} = 0$. For $k = 1$, $\widetilde{\nabla}_A\widetilde{\chi}|_{\rho=0} = 0$ for $A = 0, a$ since χ was parallel, and $\widetilde{\nabla}_\infty\widetilde{\chi}|_{\rho=0} = 0$ since it was extended to be parallel along the ρ lines.

Assume $\widetilde{\nabla}^l\widetilde{\chi}|_{\rho=0} = 0$ for $1 \le l \le k$. We must show that $\widetilde{\nabla}^{k+1}\widetilde{\chi}|_{\rho=0} = 0$ and can assume that $k \le n/2 - 2$ if n is even. We show that $\widetilde{\chi}_{\mathcal{I},A_0\cdots A_k}|_{\rho=0} = 0$ by considering various cases for the indices. First, if $A_k \neq \infty$, then the result follows by expanding out the last covariant derivative in terms of Christoffel symbols and using the induction hypothesis and the fact that ∂_{A_k} is tangent to $\{\rho = 0\}$. Next, if $A_l \neq \infty$ for some $l > 0$, then we can commute A_l all the way to the right using the above observation about symmetry in the last indices, and thus reduce to the case $A_k \neq \infty$. So we can assume all $A_l = \infty$ for $l > 0$.

If $A_0 = \infty$, then our first observation above does the job. In the following, all quantities are understood to be evaluated at $\rho = 0$, $t = 1$. If $A_0 \neq \infty$, write

$$\begin{aligned}\widetilde{\chi}_{\mathcal{I},A_0\underbrace{\infty\cdots\infty}_{k}} &= \widetilde{\chi}_{\mathcal{I},\infty A_0\underbrace{\infty\cdots\infty}_{k-1}} + \Big(\sum_{s=1}^{r}\widetilde{R}^J{}_{I_sA_0\infty}\widetilde{\chi}_{I_1\cdots I_{s-1}JI_{s+1}\cdots I_r}\Big)_{,\underbrace{\infty\cdots\infty}_{k-1}} \\ &= \sum_{s=1}^{r}\widetilde{R}^J{}_{I_sA_0\infty,\underbrace{\infty\cdots\infty}_{k-1}}\widetilde{\chi}_{I_1\cdots I_{s-1}JI_{s+1}\cdots I_r},\end{aligned} \tag{3.2}$$

where for the second equality we have used the induction hypothesis and the fact that $\widetilde{\chi}_{\mathcal{I},\infty A_0\underbrace{\infty\cdots\infty}_{k-1}} = 0$ since a differentiation index after the first is not ∞. Now the first equation in (3.1) and the skew-symmetry of the curvature tensor in the second pair of indices imply $\widetilde{R}^J{}_{I0\infty,\underbrace{\infty\cdots\infty}_{k-1}} = 0$ for all J, I and $k \ge 1$, so we obtain $\widetilde{\chi}_{\mathcal{I},0\underbrace{\infty\cdots\infty}_{k}} = 0$.

We have left only to consider $\widetilde{\chi}_{\mathcal{I},a\underbrace{\infty\cdots\infty}_{k}}$ with $1 \le a \le n$. We intend to apply Lemma 3.1 to $\widetilde{R}^J{}_{I_sa\infty,\underbrace{\infty\cdots\infty}_{k-1}}$ on the right-hand side of (3.2) for $A_0 = a$ after lowering the index J. We verify the hypothesis in Lemma 3.1 in case n is even. Denoting by L the index replacing J when it is lowered and recalling that $k \le n/2 - 2$, we have $\|LI_sa\| + 2k \le 5 + (n-4) = n + 1$. Thus the application of Lemma 3.1 is justified.

Equation (3.2), Lemma 3.1 and the induction hypothesis thus give

$$\begin{aligned}(2k+1)\widetilde{\chi}_{\mathcal{I},a\underbrace{\infty\cdots\infty}_{k}} =&(2k+1)\sum_{s=1}^{r}\widetilde{R}^{J}{}_{I_s a\infty,\underbrace{\infty\cdots\infty}_{k-1}}\widetilde{\chi}_{I_1\cdots I_{s-1}JI_{s+1}\cdots I_r}\\ =&\sum_{s=1}^{r}\widetilde{R}^{J}{}_{I_s a}{}^{p}{}_{,p\underbrace{\infty\cdots\infty}_{k-1}}\widetilde{\chi}_{I_1\cdots I_{s-1}JI_{s+1}\cdots I_r}\\ =&\Big(\sum_{s=1}^{r}\widetilde{R}^{J}{}_{I_s a}{}^{p}\widetilde{\chi}_{I_1\cdots I_{s-1}JI_{s+1}\cdots I_r}\Big)_{,p\underbrace{\infty\cdots\infty}_{k-1}}\\ =&\big(\widetilde{\chi}_{\mathcal{I},a}{}^{p}-\widetilde{\chi}_{\mathcal{I},}{}^{p}{}_{a}\big)_{,p\underbrace{\infty\cdots\infty}_{k-1}}\\ =&\widetilde{\chi}_{\mathcal{I},a}{}^{p}{}_{p\underbrace{\infty\cdots\infty}_{k-1}}-\widetilde{\chi}_{\mathcal{I},}{}^{p}{}_{ap\underbrace{\infty\cdots\infty}_{k-1}}\\ =&\widetilde{\chi}_{\mathcal{I},a}{}^{p}{}_{\underbrace{\infty\cdots\infty}_{k-1}p}-\widetilde{\chi}_{\mathcal{I},}{}^{p}{}_{a\underbrace{\infty\cdots\infty}_{k-1}p},\end{aligned}$$

where the last equality uses the observation about symmetry in last differentiated indices. Now write out each of the final covariant differentiations in the last line in terms of Christoffel symbols. Since we have shown that $\widetilde{\chi}_{\mathcal{I},A_0\cdots A_k}=0$ unless $A_1=\cdots=A_k=\infty$ and $1\le A_0\le n$, consulting with (2.5) to evaluate the Christoffel symbols gives

$$\widetilde{\chi}_{\mathcal{I},a}{}^{p}{}_{\underbrace{\infty\cdots\infty}_{k-1}p}=\partial_p\widetilde{\chi}_{\mathcal{I},a}{}^{p}{}_{\underbrace{\infty\cdots\infty}_{k-1}}+\widetilde{\Gamma}^{p}_{p0}\widetilde{\chi}_{\mathcal{I},a}{}^{0}{}_{\underbrace{\infty\cdots\infty}_{k-1}}=n\widetilde{\chi}_{\mathcal{I},a\underbrace{\infty\cdots\infty}_{k}}$$

and

$$\widetilde{\chi}_{\mathcal{I},}{}^{p}{}_{a\underbrace{\infty\cdots\infty}_{k-1}p}=\partial_p\widetilde{\chi}_{\mathcal{I},}{}^{p}{}_{a\underbrace{\infty\cdots\infty}_{k-1}}-\widetilde{\Gamma}^{\infty}_{pa}\widetilde{\chi}_{\mathcal{I},}{}^{p}{}_{\underbrace{\infty\cdots\infty}_{k}}=\widetilde{\chi}_{\mathcal{I},a\underbrace{\infty\cdots\infty}_{k}}.$$

Thus we obtain $(2k+1)\widetilde{\chi}_{\mathcal{I},a\underbrace{\infty\cdots\infty}_{k}}=(n-1)\widetilde{\chi}_{\mathcal{I},a\underbrace{\infty\cdots\infty}_{k}}$, or

$$(2k+2-n)\widetilde{\chi}_{\mathcal{I},a\underbrace{\infty\cdots\infty}_{k}}=0.$$

Hence the induction proceeds to all orders if n is odd, but we must impose $k\neq n/2-1$ if n is even. □

A consequence of the proof is the following proposition.

Proposition 3.2. *Let n be even and suppose that $\widetilde{g}$ is an infinite-order ambient metric for (M,c). (In particular (M,c) has vanishing obstruction tensor.) Let $\chi\in\Gamma\left(\otimes^r\mathcal{T}^*\right)$ satisfy $\nabla\chi=0$. If χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi}=O(\rho^{n/2})$, then χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi}=O(\rho^{\infty})$.*

Proof. Let $\widetilde{\chi}$ be the extension obtained by parallel translation along the lines $\rho \mapsto (t, x, \rho)$ as in the proof of Theorem 1.4 and let $\widetilde{\chi}'$ be an extension satisfying $\widetilde{\nabla}\widetilde{\chi}' = O(\rho^{n/2})$. Then $\widetilde{\chi}-\widetilde{\chi}' = O(\rho)$ and $\widetilde{\nabla}_\infty(\widetilde{\chi}-\widetilde{\chi}') = O(\rho^{n/2})$. Writing the latter equation in terms of ∂_ρ and Christoffel symbols and then successively differentiating with respect to ρ at $\rho = 0$ shows that $\widetilde{\chi}-\widetilde{\chi}' = O(\rho^{n/2+1})$. Hence $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$. Since the induction in the proof of Theorem 1.4 requires only $k \neq n/2 - 1$, it shows that $\widetilde{\nabla}\widetilde{\chi} = O(\rho^\infty)$. Lemma 3.1 which is used in the proof holds for $\widetilde{g}$ for all k because it just depends on homogeneity, straightness, and vanishing of the Ricci curvature. □

We remark that Theorem 1.4 implies a large collection of integrability conditions on a parallel $\chi \in \Gamma(\otimes^r \mathcal{T}^*)$. First suppose n is odd. Let $\widetilde{\chi}$ be an ambient extension of χ as in Theorem 1.4. Commuting covariant derivatives shows that $\widetilde{R}.\widetilde{\chi} = O(\rho^\infty)$, where $\widetilde{R}.$ denotes the action of $\widetilde{R}$ viewed as an element of $\Gamma(\Lambda^2 T^*\widetilde{\mathcal{G}} \otimes \operatorname{End} T\widetilde{\mathcal{G}})$, so that $\widetilde{R}.\widetilde{\chi} \in \Gamma(\Lambda^2 T^*\widetilde{\mathcal{G}} \otimes \otimes^r T^*\widetilde{\mathcal{G}})$. Further differentiating this equation and restricting to $\rho = 0$ shows that $\big((\widetilde{\nabla}^k\widetilde{R})|_{\mathcal{G}}\big).\chi = 0$, where now $(\widetilde{\nabla}^k\widetilde{R})|_{\mathcal{G}} \in \Gamma\big((\otimes^k T^*\widetilde{\mathcal{G}} \otimes \Lambda^2 T^*\widetilde{\mathcal{G}} \otimes \operatorname{End} T\widetilde{\mathcal{G}})\big|_{\mathcal{G}}\big)$ acts on χ via the $\operatorname{End} T\widetilde{\mathcal{G}}$ factor. We can regard $(\widetilde{\nabla}^k\widetilde{R})|_{\mathcal{G}}$ as a section of a weighted tensor power of a tractor bundle (see Proposition 6.5 of [FG2]), so can regard $\big((\widetilde{\nabla}^k\widetilde{R})|_{\mathcal{G}}\big).\chi = 0$ as a purely tractor equation which holds as a consequence of the fact that χ is parallel. The special case when $k = 0$ and the free $\Lambda^2 T^*\widetilde{\mathcal{G}}$ indices on $\widetilde{R}$ are tangent to $\mathcal{G}$ recovers the fact that the tractor curvature annihilates χ. When $k = 0$ and these free indices are $j\infty$, one obtains relations involving contractions of the Cotton and Bach tensors into χ. Relations of these types for special cases may be found in the literature, for example for $r = 1$ in [Go1] and for χ skew-symmetric in [Leit1], [Leit2]. In general, Theorem 1.4 is equivalent to proving the integrability conditions $\big((\widetilde{\nabla}^k\widetilde{R})|_{\mathcal{G}}\big).\chi = 0$ for all k. A similar discussion holds for n even, keeping in consideration that the number of differentations transverse to $\mathcal{G}$ is restricted.

4. Critical order for n even

In this section we consider some examples illustrating what can happen concerning ambient extension of parallel tractors at the critical order in even dimensions. The main issue is that in general an ambient metric is no longer determined to higher order solely by the conformal structure; there is an ambiguity at order $n/2$. So whether or not a given parallel tractor has a parallel ambient extension can depend on which ambient metric is chosen. The natural general framework for such investigations would be to consider ambient metrics with log terms as in Theorem 3.10 of [FG2]. In this case parallel extensions of tractors must be expected to have expansions with log terms as well. For simplicity we generally restrict consideration here to the case of conformal structures with vanishing obstruction tensor, for which log terms do not enter. The one exception is that our discussion of Fefferman metrics of nondegenerate hypersurfaces in $\mathbb{C}^n$ applies also to some ambient metrics with log terms.

The dependence of the covariant derivatives of the curvature tensor of an ambient metric on the ambiguity plays an important role in these considerations. Let $n \geq 4$ be even, let (M, c) be a conformal structure with vanishing obstruction tensor, and let $\widetilde{g}$ be an infinite-order ambient metric for (M, c) in normal form relative to a representative $g \in c$. Recall from §2 that $\widetilde{g}$ takes the form (2.1) and tf $\left(\partial_\rho^{n/2} g_\rho|_{\rho=0}\right)$ is the ambiguity. Define the strength of lists of indices $\|IJ\cdots K\|$ as in Lemma 3.1. Proposition 6.2 of [FG2] shows that if $\|IJKLM_1\cdots M_r\| \leq n+1$, then the restriction to $\rho = 0$, $t = 1$ of $\widetilde{R}_{IJKL,M_1\cdots M_r}$ is independent of the ambiguity and defines a natural tensor invariant of g as the indices between 1 and n vary; in fact it can be expressed as a linear combination of contractions $\operatorname{contr}(\nabla^{m_1} R \otimes \cdots \otimes \nabla^{m_N} R)$ of the covariant derivatives of the curvature tensor of g such that $2N + 2 + \sum_{i=1}^N m_i = \|IJKLM_1\cdots M_r\|$. On the other hand, if $\|IJKLM_1\cdots M_r\| \geq n+2$, then $\widetilde{R}_{IJKL,M_1\cdots M_r}|_{\rho=0,\,t=1}$ may depend on the ambiguity. Proposition 6.6 of [FG2] shows that the component $\widetilde{R}_{\infty ij\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ parametrizes the ambiguity: any two infinite-order ambient metrics in normal form for which this component agree must agree to infinite order, and this component can be prescribed to be an arbitrary trace-free symmetric 2-tensor on M subject to the condition that its divergence is a particular natural 1-form invariant of the initial metric g. Another treatment of similar considerations concerning the ambiguity is contained in [ČG1].

Let now $\chi \in \Gamma(\otimes^r \mathcal{T}^*)$ be parallel for (M, c). We are interested in the questions of existence and uniqueness of infinite-order ambient metrics $\widetilde{g}$ for which χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^\infty)$. Proposition 3.2 shows that for a given $\widetilde{g}$, χ has such an extension if and only if it has an extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$. We will see that for some χ there is no such extension for any choice of $\widetilde{g}$, for some χ there is such an extension for precisely one $\widetilde{g}$, and for some χ such extensions exist for infinitely many choices of $\widetilde{g}$.

A trivial example of a parallel tractor in even dimensions which has a parallel ambient extension for more than one infinite-order ambient metric is the tractor metric. Clearly for any ambient metric, the ambient metric itself provides a parallel extension. If one index is raised, the identity endomorphism of the tractor bundle has as a parallel extension the identity endomorphism of the ambient tangent bundle, and in this realization the parallel extension is actually independent of which ambient metric is chosen.

We begin by formulating a condition on χ under which there is at most one choice of infinite-order ambient metric with respect to which χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$. Recall that a 1-form on M can be inserted as the injecting part of an adjoint tractor. (An adjoint tractor is a section of $\operatorname{End}\mathcal{T}$ which is skew with respect to h.) Namely, if $\eta \in T_x^*M$ then $\pi^*\eta$ defines a section of $T^*\mathcal{G}|_{\mathcal{G}_x}$ which annihilates T. Since T_I spans the annihilator of $T\mathcal{G} \subset T\widetilde{\mathcal{G}}$, we may regard $\pi^*\eta$ as

an equivalence class of sections of $T^*\widetilde{\mathcal{G}}|_{\mathcal{G}_x}$ defined modulo T_I and homogeneous of degree 0 with respect to the δ_s, and therefore as an equivalence class of elements of $\mathcal{T}^*_x(-1)$ defined modulo T_I. So we can define a bundle map $\mathbb{I} : T^*M \to \mathrm{End}_{\mathrm{skew}}\,\mathcal{T}$ by $\mathbb{I}(\eta)^J{}_I = 2h^{JK}T_{[K}(\pi^*\eta)_{I]}$, where h^{JK} denotes the inverse tractor metric. With respect to the splitting induced by any representative g one has $\mathbb{I}(\eta)^0{}_i = \eta_i$, $\mathbb{I}(\eta)^j{}_\infty = -\eta^j$, other $\mathbb{I}(\eta)^J{}_I = 0$. Now let $\chi \in \Gamma\left(\otimes^r\mathcal{T}^*\right)$. Define a bundle map $F_\chi : T^*M \to \otimes^r\mathcal{T}^*$ by $F_\chi(\eta) = \mathbb{I}(\eta).\chi$, where the . refers to the action of $\mathrm{End}\,\mathcal{T}$ on $\otimes^r\mathcal{T}^*$.

Definition 4.1. Let $\chi \in \Gamma\left(\otimes^r\mathcal{T}^*\right)$. We say that χ is *determining* if the induced map $F_\chi : \Gamma(T^*M) \to \Gamma(\otimes^r\mathcal{T}^*)$ is injective.

Note that we are requiring that the induced map on smooth sections be injective, not that F_χ is injective as a bundle map. Observe that the tractor metric h is not determining since $K.h = 0$ for any adjoint tractor K.

Proposition 4.2. *Let (M, c) be a conformal manifold of even dimension $n \geq 4$ with vanishing obstruction tensor. Let $\chi \in \Gamma\left(\otimes^r\mathcal{T}^*\right)$ be parallel and determining. Then up to infinite order and up to diffeomorphism, there is at most one infinite-order ambient metric with respect to which χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$.*

Proof. Pick $g \in c$ and assume that $\widetilde{g}$ is in normal form with respect to g. If $\widetilde{\chi}$ is an extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$, then at $\rho = 0$, $t = 1$ we have (notation as in the proof of Theorem 1.4):

$$\begin{aligned}
0 =&\widetilde{\chi}_{\mathcal{I},a\underbrace{\infty\cdots\infty}_{(n-2)/2}} - \widetilde{\chi}_{\mathcal{I},\infty a\underbrace{\infty\cdots\infty}_{(n-4)/2}} \\
=&\Big(\sum_{s=1}^{r}\widetilde{R}^J{}_{I_s a\infty}\widetilde{\chi}_{I_1\cdots I_{s-1}JI_{s+1}\cdots I_r}\Big)_{,\underbrace{\infty\cdots\infty}_{(n-4)/2}} \qquad (4.1)\\
=&\sum_{s=1}^{r}\widetilde{R}^J{}_{I_s a\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}\widetilde{\chi}_{I_1\cdots I_{s-1}JI_{s+1}\cdots I_r}.
\end{aligned}$$

Of the components $\widetilde{R}^J{}_{Ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ which enter this equation, only $\widetilde{R}^0{}_{ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ and $\widetilde{R}^j{}_{\infty a\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ depend on the ambiguity of the ambient metric. All other components are determined by g. Choose a vector field v on M. Define a 1-form η on M by $\eta_i = v^a\widetilde{R}_{\infty ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$. Then $\mathbb{I}(\eta)^J{}_I = v^a\widetilde{R}^J{}_{Ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ if ${}^J{}_I = {}^0{}_i$ or ${}^J{}_I = {}^j{}_\infty$ and $\mathbb{I}(\eta)^J{}_I = 0$ otherwise. If we set $D^J{}_I = v^a\widetilde{R}^J{}_{Ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}} - \mathbb{I}(\eta)^J{}_I$, then $D^J{}_I$ is independent of the ambiguity in the ambient metric. Now the contraction of v^a with (4.1) can be written $F_\chi(\eta) = -D.\chi$. By the injectivity of F_χ on $\Gamma(T^*M)$, it follows that for each v there is

at most one possibility for η. Therefore there is at most one possibility for the tensor $\widetilde{R}_{\infty ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$. But this tensor determines the ambiguity in the ambient metric. □

Let us consider the case $r = 1$ so that χ is a parallel section of $\mathcal{T}^*$, which we assume is nonzero. We sometimes call such χ a parallel tractor 1-form. In the terminology of Gover [Go3], χ defines an almost Einstein structure. There is a large literature concerning parallel tractor 1-forms, especially because of their relation to conformally Einstein metrics and their basic role in conformal holonomy.

Upon writing the equation $\nabla\chi = 0$ in the splitting corresponding to a metric g using (2.6), one finds that χ is determined by χ_0, and that χ_0 must satisfy the equation $\operatorname{tf}\big((\nabla^2_{ij} + P_{ij})\chi_0\big) = 0$. Specifically, writing $\sigma = \chi_0$, χ is given by

$$\chi = \begin{pmatrix}\chi_0\\ \chi_i\\ \chi_\infty\end{pmatrix} = \begin{pmatrix}\sigma\\ \sigma_i\\ -\frac{1}{n}(\Delta\sigma + J\sigma)\end{pmatrix}$$

where $\Delta = \nabla^k\nabla_k$ and $J = R/2(n-1)$. In this way parallel sections of $\mathcal{T}^*$ are in one-to-one-correspondence with solutions to the conformally invariant equation

$$\operatorname{tf}\big((\nabla^2_{ij} + P_{ij})\sigma\big) = 0. \tag{4.2}$$

Since χ is parallel and nonzero it is nonvanishing, so the 2-jet of σ is nonvanishing. In particular, $\{\sigma \neq 0\}$ is open and dense. The conformal transformation law for the trace-free Schouten tensor shows that on this set the rescaled metric $\sigma^{-2}g$ is Einstein. If $\Sigma := \{\sigma = 0\}$ is nonempty, the conformal structure extends smoothly across Σ but the Einstein representative $\sigma^{-2}g$ becomes singular. Since the obstruction tensor vanishes for conformal classes containing an Einstein metric, by continuity it vanishes for even-dimensional conformal classes admitting a nonzero parallel tractor 1-form.

Proposition 4.3. *Let $\chi \in \Gamma(\mathcal{T}^*)$ be nonzero and satisfy $\nabla\chi = 0$. Then χ is determining.*

Proof. We have $\big(F_\chi(\eta)\big)_I = -\mathbb{I}(\eta)^J{}_I\chi_J$. In particular, $\big(F_\chi(\eta)\big)_i = -\eta_i\chi_0$. Since $\chi_0 \neq 0$ on an open dense set, it follows that F_χ is injective on $\Gamma(T^*M)$. □

Theorem 1.4 shows that if n is odd, then any parallel tractor 1-form has an ambient extension parallel to infinite order (with respect to the unique ambient metric up to infinite order and diffeomorphism), and the same is true to order $n/2 - 1$ if n is even. Propositions 4.2 and 4.3 imply that for n even, there is at most one determination of the ambiguity in an infinite-order ambient metric with respect to which χ has an ambient extension parallel to infinite order. We analyze the question in this case of whether there exists a choice of the ambiguity with respect to which χ has an ambient extension parallel to infinite order.

Consider first the situation near the set $\{\sigma \neq 0\}$; equivalently consider a conformal class containing a chosen Einstein metric. In this case in all dimensions one can

identify explicitly an ambient metric for which there is an explicit parallel extension of χ. If g is Einstein and we set $\lambda = J/n = R/2n(n-1)$, then $\widetilde{g}$ defined by (2.1) with $g_\rho = (1+\lambda\rho)^2 g$ satisfies $\mathrm{Ric}(\widetilde{g}) = 0$. If n is odd then this is to infinite order the unique ambient metric in normal form relative to g. However, if n is even then there are other infinite-order ambient metrics corresponding to other choices of the ambiguity. Proposition 7.5 of [FG2] shows that for n even this choice is canonical in the sense that up to infinite order and up to diffeomorphism, it is independent of which Einstein metric in the conformal class is chosen (usually there is only one up to homothety, but there may be others). Henceforth, if g is Einstein we will denote by $\widetilde{g}^c$ the ambient metric given by (2.1) with

$$g_\rho = (1+\lambda\rho)^2 g. \tag{4.3}$$

If g is Einstein, the corresponding solution of (4.2) is $\sigma = 1$. Thus the parallel tractor is given in the associated Einstein scale by $\chi = (1, 0, -\lambda)$. It is straightforward to verify using (2.5) with $g_\rho = (1+\lambda\rho)^2 g$ that a parallel extension of χ for the ambient metric $\widetilde{g}^c$ is given by $\widetilde{\chi} = (1-\lambda\rho)dt - t\lambda\, d\rho = d[t(1-\lambda\rho)]$. We conclude

Proposition 4.4. *Let $n \geq 4$ be even and suppose that (M, c) is a conformal class containing an Einstein metric g. Up to infinite order and up to diffeomorphism, the canonical ambient metric $\widetilde{g}^c$ is the unique infinite-order ambient metric for (M, c) with respect to which the associated parallel tractor 1-form χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$.*

We remark that [Leit1] for $\lambda \neq 0$ and [Leis] for $\lambda = 0$ have shown that $\widetilde{g}^c$ can be written in a way that makes it evident that any parallel tractor has a parallel ambient extension (in fact even that the holonomy of the tractor connection equals the holonomy of $\widetilde{g}^c$). Namely, if $\lambda \neq 0$, set $u = (1+\lambda\rho)t$, $v = (1-\lambda\rho)t$. Then

$$\widetilde{g}^c = 2d(\rho t)dt + t^2(1+\lambda\rho)^2 g = \frac{1}{2\lambda}(du^2 - dv^2) + u^2 g$$

is translation-invariant in v relative to the (u, v, x) coordinates and ∂_v is parallel. Thus a parallel extension of any parallel tractor is obtained by extending it to be translation-invariant in v. When $\lambda = 0$, the same reasoning applies upon setting $v = \rho t$ so that $\widetilde{g}^c = 2dvdt + t^2 g$ is translation-invariant in v relative to (t, v, x) coordinates and ∂_v is parallel. When combined with Proposition 4.2, this gives another proof that for n even, the canonical ambient metrics associated to two different Einstein metrics in the same conformal class are related by a diffeomorphism to infinite order (Proposition 7.5 of [FG2]). Namely, by the Leitner/Leistner result, the parallel tractor associated to the second Einstein metric has a parallel ambient extension relative to the canonical ambient metric for the first Einstein metric, so by Proposition 4.2, the two canonical ambient metrics are diffeomorphic to infinite order.

A basic class of almost Einstein structures for which Σ is nonempty consists of those determined by Poincaré-Einstein metrics. We will give necessary and sufficient

conditions on an even-dimensional Poincaré-Einstein metric in order that the associated parallel tractor 1-form admit an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$ for some choice of infinite-order ambient metric. In particular, this will give examples of parallel tractor 1-forms which do not admit such an ambient extension for any infinite-order ambient metric.

Let $\Sigma \subset M$ be an embedded hypersurface. In this paper we will say that a metric g_+ on $M \setminus \Sigma$ is Poincaré-Einstein if $\mathrm{Ric}(g_+) = -(n-1)g_+$ and if $g_+ = r^{-2}g$ for some defining function r for Σ and smooth metric g on M of signature (p,q) such that $g|_{T\Sigma}$ has signature $(p-1,q)$. (In particular, since g is assumed smooth, the obstruction tensor of $g|_{T\Sigma}$ vanishes if $n \geq 5$ is odd.) We could alternately take g_+ to satisfy $\mathrm{Ric}(g_+) = (n-1)g_+$ and require $g|_{T\Sigma}$ to have signature $(p,q-1)$; the two formulations are equivalent under the change $g_+ \mapsto -g_+$, $(p,q) \mapsto (q,p)$.

The conformal structure (M,c) with $c = [g]$ is determined by g_+ and admits a parallel tractor 1-form given relative to g by

$$\chi = \begin{pmatrix} \chi_0 \\ \chi_i \\ \chi_\infty \end{pmatrix} = \begin{pmatrix} r \\ r_i \\ -\frac{1}{n}(\Delta r + Jr) \end{pmatrix}. \tag{4.4}$$

Possibly rescaling r and g, one can identify a neighborhood of Σ in M with a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$ so that r is the variable in $\mathbb{R}$ and g takes the form $g = dr^2 + h_r$ for a smooth 1-parameter family h_r of metrics on Σ. If n is odd, then the Taylor expansion of h_r in r is even to infinite order. If $n \geq 4$ is even, then h_r is even in r to order $n-2$, but a typical Poincaré-Einstein metric has $\partial_r^{n-1} h_r|_\Sigma \neq 0$. (See [FG2].) For n even, we say that g_+ is even if $\partial_r^{n-1} h_r|_\Sigma = 0$. This is independent of the choices, and the Taylor expansion of h_r is then even to infinite order. If n is odd, all Poincaré-Einstein metrics will be said to be even.

Proposition 4.5. *Let $n \geq 4$ be even and let g_+ be a Poincaré-Einstein metric with associated parallel tractor χ. There exists an infinite-order ambient metric for (M,c) with respect to which χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$ if and only if g_+ is even.*

Proof. Write $g = dr^2 + h_r$ near Σ as above. Note first that since $|dr|_g^2 = 1$, the conformal transformation law $P_{g_+} = P_g + r^{-1}\nabla^2 r - \frac{1}{2}|dr|_g^2\, g_+$ of the Schouten tensor and the Einstein condition on g_+ imply that $\nabla^2 r + P_g r = 0$. Taking the trace gives $\Delta r + Jr = 0$, so $\chi_\infty = 0$.

Let $\widetilde{g}$ be an infinite-order ambient metric for (M,c) in normal form relative to g and let $\widetilde{\chi}$ be the extension given in the proof of Theorem 1.4 which satisfies $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{(n-2)/2})$. The first steps in the induction in Theorem 1.4 hold also at the next order, so we know that $\widetilde{\chi}_{I,\infty a \underbrace{\infty\cdots\infty}_{(n-4)/2}} = 0$ at $\rho = 0$, $t = 1$, and that χ has an extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$ if and only if $\widetilde{\chi}_{I,a \underbrace{\infty\cdots\infty}_{(n-2)/2}} = 0$ at $\rho = 0$, $t = 1$. All succeeding

expressions are understood to be evaluated at $\rho = 0$, $t = 1$. Equation (3.2) thus shows that χ has an extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$ if and only if $\widetilde{R}^J{}_{Ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}\widetilde{\chi}_J = 0$; that is if and only if (recall $\chi_\infty = 0$)

$$\widetilde{R}_{\infty Ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}r = -\widetilde{R}^j{}_{Ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}r_j. \tag{4.5}$$

The right-hand side of (4.5) for $I = i$ is independent of the ambiguity of the ambient metric; it is determined by g alone. Using the expression for the leading terms in the ambient metric expansion (see (3.21) of [FG2]), it is not hard to see that the tensor $\widetilde{R}_{jia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ which appears on the right-hand side of (4.5) for $I = i$ has the form

$$\widetilde{R}_{jia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}} = c\Delta^{(n-4)/2}C_{aij} + \Lambda_{aij}, \tag{4.6}$$

where $C_{aij} = P_{ai,j} - P_{aj,i}$ denotes the Cotton tensor of g, $c \neq 0$, and Λ_{aij} is a linear combination of contractions of covariant derivatives of curvature of g which are at least quadratic and involve at most $n-5$ derivatives of curvature.

Label objects on Σ by Greek letters α, β and let a $\diamondsuit$ index correspond to r in the identification $M \sim \Sigma\times\mathbb{R}$, so that $i \leftrightarrow (\alpha,\diamondsuit)$. Denote ∂_r by $'$. We first show that the right-hand side of (4.5) for $I = \beta$ and $a = \alpha$ vanishes on Σ if and only if g_+ is even. By (4.6) we have

$$\widetilde{R}^j{}_{\beta\alpha\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}r_j = \widetilde{R}_{\diamondsuit\beta\alpha\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}} = c\Delta^{(n-4)/2}C_{\alpha\beta\diamondsuit} + \Lambda_{\alpha\beta\diamondsuit}.$$

Now $\Lambda_{\alpha\beta\diamondsuit}$ can be written as a linear combination of contractions $\operatorname{contr}(\nabla^{m_1}R\otimes\cdots\otimes\nabla^{m_N}R)$ with three free indices α, β, $\diamondsuit$. The contractions can be expanded corresponding to the block diagonal form of g. In each term, at least one of the $\nabla^{m_i}R$ will have an odd number of $\diamondsuit$ indices. Since h_r is even to order $n-2$ and Λ involves at most $n-3$ derivatives of g, this component $\nabla^{m_i}R$ vanishes on Σ. Thus $\Lambda_{\alpha\beta\diamondsuit} = 0$ on Σ. Similar analysis considering also the leading nonzero term shows that

$$\Delta^{(n-4)/2}C_{\alpha\beta\diamondsuit} = c'\partial_r^{n-1}h_{\alpha\beta} \quad \text{on } \Sigma$$

where $c' \neq 0$. Thus the right-hand side of (4.5) for $I = \beta$, $a = \alpha$ vanishes on Σ if and only if $\partial_r^{n-1}h_{\alpha\beta} = 0$ on Σ, that is if and only if g_+ is even.

It follows immediately that if χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$ then g_+ is even, since the left-hand side of (4.5) clearly vanishes on Σ.

Next we prove the converse. Away from Σ, χ is the parallel tractor associated to the Einstein metric g_+. We have seen in this case in Proposition 4.4 that there is a unique determination of the ambiguity in the ambient metric so that χ has an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$. Thus away from Σ there is a unique determination

of $\widetilde{R}_{\infty ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ for which (4.5) holds. We will show that if g_+ is even, then the so-determined tensor $\widetilde{R}_{\infty ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ extends smoothly across Σ. The desired result follows immediately. This tensor is trace-free and satisfies the divergence constraint on Σ relative to g by continuity, so it determines an infinite-order ambient metric. All quantities appearing in (4.5) are then smooth across Σ, so (4.5) holds on Σ by taking limits. (If g_+ is not even, the tensor $\widetilde{R}_{\infty ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ uniquely determined away from Σ blows up on approach to Σ.)

To see that $\widetilde{R}_{\infty ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ extends smoothly across Σ, first take $I=\infty$ in (4.5) to obtain $\widetilde{R}_{\infty\diamondsuit a\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}=0$. So the components $\widetilde{R}_{\infty ia\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ with i or $a=\diamondsuit$ certainly extend smoothly across Σ; they vanish identically nearby. We have seen that if g_+ is even, then the right hand side of (4.5) for $I=\beta$, $a=\alpha$ extends smoothly across Σ and vanishes on Σ. Thus dividing by r shows that $\widetilde{R}_{\infty\beta\alpha\infty,\underbrace{\infty\cdots\infty}_{(n-4)/2}}$ extends smoothly across Σ as well. □

If n is even and g_+ is an even Poincaré-Einstein metric, then the ambient metric of Proposition 4.5 with respect to which χ has an ambient extension parallel to infinite order is uniquely determined to infinite order up to diffeomorphism by Propositions 4.2 and 4.3. It may therefore be regarded as a distinguished ambient metric for the conformal class c determined by g_+. Proposition 4.4 shows that away from Σ it agrees to infinite order up to diffeomorphism with the canonical ambient metric determined by the Einstein metric $g_+\in c|_{M\setminus\Sigma}$ in the sense of our previous discussion.

We conclude our discussion of Poincaré-Einstein metrics by showing that in all dimensions, one can identify explicitly the parallel ambient extension of the parallel tractor 1-form (4.4) corresponding to an even Poincaré-Einstein metric (as well as the ambient metric with respect to which it is parallel). This is closely related to Theorem 6.3 of [Go3] and uses a formula of Juhl [J] for a Poincaré-Einstein metric whose conformal infinity is $g=dr^2+h_r$, where $g_+=r^{-2}g$ is an even Poincaré-Einstein metric. Recall that we defined a Poincaré-Einstein metric to be even if $g_+=r^{-2}(dr^2+h_r)$ where the Taylor expansion of h_r is even in r to infinite order, and this holds for all Poincaré-Einstein metrics if n is odd. If g_+ is even and positive definite, Biquard's unique continuation theorem ([B]) shows that $h_r=h_{-r}$ for r near 0. But we do not know if this holds for other signatures. For simplicity of exposition, in the subsequent discussion we will restrict attention to even Poincaré-Einstein metrics for which $h_r=h_{-r}$. This is no real loss of generality: one can construct two such metrics from any even Poincaré-Einstein metric by reflecting across $r=0$ the restriction of h_r to either $r>0$ or $r<0$.

Let $n \geq 3$ and let $g_+ = r^{-2}g$ be an even Poincaré-Einstein metric with $g = dr^2 + h_r$ where $h_r = h_{-r}$. Define k_u for $u \geq 0$ small by $k_u = h_{\sqrt{u}}$ so that k_u is smooth up to $u = 0$ and $h_r = k_{r^2}$ for all r near 0. Juhl has discovered (Theorem 7.2 of [J]) that the metric g_{++} given by

$$g_{++} = s^{-2}\big(ds^2 + dr^2 + k_{r^2+s^2}\big) \tag{4.7}$$

is a Poincaré-Einstein metric with conformal infinity $[g]$ in normal form relative to g. Juhl proves this by a direct calculation that g_{++} satisfies $\mathrm{Ric}(g_{++}) = -ng_{++}$ for $s \neq 0$. An alternate proof (which could be used to guess the result) is to begin with the Einstein metric g_+ in the conformal class $[g]$ away from Σ. The formula for the "canonical" Poincaré metric associated to an Einstein metric (see (7.13) of [FG2]; this is the Poincaré metric analogue of (4.3) above) gives

$$g_{++} = s^{-2}\left[ds^2 + (1 + \tfrac{1}{4}s^2)^2 g_+\right]. \tag{4.8}$$

Now g_{++} can be put into normal form relative to the metric $g = dr^2 + h_r$ in the conformal class. This can be carried out explicitly: substituting $g_+ = r^{-2}(dr^2 + k_{r^2})$ in (4.8) and making the change of variables

$$r = \sqrt{r'^2 + s'^2} \qquad\qquad s = 2\frac{\sqrt{r'^2 + s'^2} - r'}{s'}$$

with inverse

$$r' = \frac{4 - s^2}{4 + s^2}\, r \qquad\qquad s' = \frac{sr}{1 + \frac{1}{4}s^2}$$

one obtains without difficulty $g_{++} = s'^{-2}\big(ds'^2 + dr'^2 + k_{r'^2+s'^2}\big)$. Relabeling the variables gives (4.7).

The fact that g_+ is even is of course not used in verifying that g_{++} defined by (4.7) is Einstein. But if g_+ is not even, then g_{++} has as conformal infinity $[dr^2 + h_{|r|}]$, which is not smooth across $\{r = 0\}$ and does not agree with $[g]$ for $r < 0$.

Recall that in normal form, an ambient metric and the associated even Poincaré metric are related by the change of variable $s^2 = -2\rho$; see Chapter 4 of [FG2]. Thus the metric $\widetilde{g}_+$ defined by (2.1) with $g_\rho = dr^2 + k_{r^2-2\rho}$ satisfies $\mathrm{Ric}(\widetilde{g}_+) = 0$ for $\rho < 0$. (In fact, this holds where g_ρ makes sense, i.e. for $2\rho < r^2$.) If we choose some smooth extension of k_u to $\{u < 0\}$ and define g_ρ by the same formula, then $\widetilde{g}_+$ is defined in a neighborhood of $\{\rho = 0\}$ and is an ambient metric (to infinite order in any dimension) for $[g]$ in normal form relative to g.

Proposition 4.6. *Suppose $n \geq 3$ and let $g_+ = r^{-2}g$ be an even Poincaré-Einstein metric, where $g = dr^2 + k_{r^2}$. Let χ be the associated parallel tractor 1-form given by (4.4). Let $\widetilde{g}_+$ be the ambient metric defined by (2.1) with $g_\rho = dr^2 + k_{r^2-2\rho}$ as above. Then the 1-form $\widetilde{\chi} = rdt + tdr = d(rt)$ is a parallel ambient extension of χ. ($\widetilde{\chi}$ is independent of ρ and has zero $d\rho$ component.)*

Proof. We have observed at the beginning of the proof of Proposition 4.5 that $\chi_\infty = 0$, so $\widetilde{\chi}$ is certainly an ambient extension of χ. The condition $\widetilde{\nabla}\widetilde{\chi} = 0$ becomes $\widetilde{\nabla}^2(rt) = 0$ and is a straightforward verification from (2.5); the function rt is a quadratic polynomial in the coordinates. The verification uses only that g_ρ has the form $dr^2+k_{r^2-2\rho}$ for some 1-parameter family of metrics k on Σ; the Einstein condition does not enter. □

Proposition 4.6 of course gives an alternate proof of the existence of an ambient extension of χ parallel to infinite order (Theorem 1.4 and Proposition 4.5) for even Poincaré-Einstein metrics. It also identifies explicitly the distinguished ambient metric associated to g_+ defined immediately after the proof of Proposition 4.5. Finally, it gives an explicit realization of Theorem 6.3 of [Go3]: by the relation again between ambient metrics and Poincaré metrics in normal form, the ambient metric in normal form relative to the metric $h_0 = g|_{T\Sigma}$ on Σ is $\widetilde{g} = 2\rho dt^2+2tdtd\rho+t^2k_{-2\rho} = \widetilde{g}_+|_{r=0,dr=0}$.

Consider next the case of tractor 2-forms. Again there is much literature; see e.g. [Leit1], [Leit2], [Go2], [H]. A section $\chi \in \Gamma(\Lambda^2\mathcal{T}^*)$ has components $\chi_{0j} \in \Gamma(T^*M)$, $\chi_{ij} \in \Gamma(\Lambda^2T^*M)$, $\chi_{0\infty} \in C^\infty(M)$, $\chi_{j\infty} \in \Gamma(T^*M)$ with respect to the splitting determined by a choice of representative g. Just as for tractor 1-forms, writing the equation $\nabla\chi = 0$ in terms of components using the tractor Christoffel symbols (2.6) shows that a parallel χ is determined by its projecting part χ_{0j}, and the projecting part satisfies the conformal Killing equation. Set $\alpha_j = \chi_{j0} = -\chi_{0j}$. If $\nabla\chi = 0$, then α satisfies $\alpha_{(i,j)} = \frac{1}{n}\alpha_{k,}{}^k g_{ij}$ and one has

$$\chi_{ij} = \alpha_{[i,j]}, \qquad \chi_{0\infty} = \tfrac{1}{n}\alpha_{k,}{}^k, \qquad \chi_{j\infty} = \tfrac{1}{n}\alpha_{k,}{}^k{}_j + P_j{}^k\alpha_k. \tag{4.9}$$

If χ is not identically 0 then it is nonvanishing, so the 2-jet of α is nonvanishing and $\{\alpha \neq 0\}$ is dense.

Proposition 4.7. *Let $\chi \in \Gamma(\Lambda^2\mathcal{T}^*)$. If $\alpha_k\alpha^k \neq 0$ on a dense set, then χ is determining.*

Proof. We have $F_\chi(\eta)_{IJ} = -\mathbb{I}(\eta)^K{}_I\chi_{KJ} - \mathbb{I}(\eta)^K{}_J\chi_{IK}$, so $F_\chi(\eta)_{ij} = 2\eta_{[i}\alpha_{j]}$. Thus if $F_\chi(\eta) = 0$, then on the set where $\alpha \neq 0$ we have $\eta = c\alpha$ for a smooth function c. Now $F_\chi(\eta)_{0\infty} = 0$ gives $\eta^k\alpha_k = 0$, so it follows that $c = 0$ where $\alpha^k\alpha_k \neq 0$. Hence $\eta = 0$ on a dense set, so $\eta = 0$ everywhere since it is smooth. □

In particular, if the signature is definite, every nonzero parallel tractor 2-form is determining.

On the other hand, for the null case we have

Proposition 4.8. *Let $\chi \in \Gamma(\Lambda^2\mathcal{T}^*)$ satisfy $\nabla\chi = 0$. If $\alpha_k\alpha^k$ vanishes identically, then χ is not determining.*

Proof. We can assume that α is not identically zero. It suffices to show that $F_\chi(\alpha) = 0$. We have $F_\chi(\eta)_{j0} = 0$ for any η. The argument in the proof of Proposition 4.7 shows

that $F_\chi(\alpha)_{ij} = 0$ and $F_\chi(\alpha)_{0\infty} = 0$. It remains to show that $F_\chi(\alpha)_{j\infty} = 0$. Using (4.9) we have

$$-F_\chi(\alpha)_{j\infty} = \alpha_j\chi_{0\infty} - \alpha^k\chi_{jk} = \tfrac{1}{n}\alpha_j\alpha_{k,}{}^k - \alpha^k\alpha_{[j,k]} = \tfrac{1}{n}\alpha_j\alpha_{k,}{}^k - \tfrac{1}{2}\alpha^k\alpha_{j,k} + \tfrac{1}{2}\alpha^k\alpha_{k,j}.$$

Now $\alpha^k\alpha_{k,j} = 0$ since α is null. Contracting the conformal Killing equation with α then gives $\alpha^k\alpha_{j,k} = \frac{2}{n}\alpha_j\alpha_{k,}{}^k$, so $F_\chi(\alpha)_{j\infty} = 0$ as desired. □

Proposition 4.8 suggests that parallel tractor 2-forms with null projecting part are candidates for existence of parallel ambient extensions for more than one infinite-order ambient metric. We show next that this happens for Fefferman conformal structures associated to nondegenerate hypersurfaces in $\mathbb{C}^n$.

In [Leit3] and [ČG2], it was shown that that Fefferman conformal structures of nondegenerate integrable CR manifolds of hypersurface type are locally characterized by the existence of a parallel almost complex structure $\mathbb{J}^I{}_J$ on $\mathcal{T}$ such that $\chi_{IJ} := \mathbb{J}_{IJ}$ is a tractor 2-form, where the index is lowered with the tractor metric. Equivalently they are characterized by the existence of a parallel tractor 2-form χ such that raising an index gives an almost complex structure. The underlying manifold M of the Fefferman conformal structure is a circle bundle over the CR manifold and $\mathbb{J}$ is determined by the property that the vector field on M given by $\mathbb{J}T \mod T$ is the infinitesimal S^1 action. This makes sense because $\mathbb{J}T$ is a section of $T\widetilde{\mathcal{G}}|_{\mathcal{G}}$ homogeneous of degree 0 and orthogonal to T with respect to the tractor metric, so it projects to a vector field on M. The fact that $\mathbb{J}$ is determined by this projection is just the statement that χ is determined by its projecting part α, which we have seen in (4.9).

In Fefferman's original construction the CR manifold is a nondegenerate real hypersurface $\mathcal{M}$ in $\mathbb{C}^n$, $n \geq 2$, in which case the circle bundle $M = S^1 \times \mathcal{M}$ is trivial. Fefferman showed in [F] that there is a smooth defining function u for $\mathcal{M}$ uniquely determined mod $O(u^{n+2})$ such that $J(u) = 1 + O(u^{n+1})$, where

$$J(u) = (-1)^{r+1}\det\begin{pmatrix} u & u_{\bar{j}} \\ u_i & u_{i\bar{j}} \end{pmatrix}_{1\leq i,j\leq n}.$$

That $\mathcal{M}$ is nondegenerate means that the Levi form $-u_{i\bar{j}}|_{T^{1,0}\mathcal{M}}$ is nondegenerate, and we have taken its signature to be (r,s), $r+s = n-1$. Set $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. A representative g for the conformal structure is by definition the pullback to

$$S^1\times\mathcal{M} = \{(z^0, z) : |z^0| = 1,\ z\in\mathcal{M}\} \subset \mathbb{C}^*\times\mathbb{C}^n$$

of the Kähler metric $\widetilde{g}$ defined on a neighborhood $\widetilde{\mathcal{G}}$ of $\mathbb{C}^*\times\mathcal{M}$ in $\mathbb{C}^*\times\mathbb{C}^n$ by

$$\widetilde{g} = \partial^2_{\alpha\overline{\beta}}(-|z^0|^2u)dz^\alpha d\overline{z}^\beta. \tag{4.10}$$

The metric bundle $\mathcal{G}$ can be identified with $\mathbb{C}^*\times\mathcal{M}$ and the dilations on $\widetilde{\mathcal{G}}$ are just the usual dilations on the $\mathbb{C}^*$ factor. Now $\widetilde{g}$ is clearly homogeneous of degree 2 and it satisfies the initial condition $\iota^*\widetilde{g} = \mathbf{g}_0$ by the definition of g. It is easily checked that

$\widetilde{g}$ is straight. Its Ricci curvature

$$\partial^2_{\alpha\overline{\beta}}(\log|\det g_{\rho\overline{\sigma}}|)dz^\alpha d\overline{z}^\beta = \partial^2_{i\overline{j}}(\log J(u))dz^i d\overline{z}^j \tag{4.11}$$

is clearly $O(u^{n-1})$ and is easily seen to be $O^+_{IJ}(\rho^{n-1})$. So $\widetilde{g}$ is an ambient metric for $(S^1 \times \mathcal{M}, [g])$.

Fix a defining function u satisfying $J(u) = 1 + O(u^{n+1})$. Theorem 2.11 of [Gr2] shows that for each $a \in C^\infty(\mathcal{M})$, there is a v uniquely determined to infinite order having an asymptotic expansion of the form

$$v \sim u \sum_{k=0}^{\infty} \eta_k (u^{n+1} \log|u|)^k \tag{4.12}$$

with each η_k smooth, such that $\eta_0 = 1 + au^{n+1} + O(u^{n+2})$ and $J(v) = 1$ to infinite order. For such v, $\widetilde{g}$ defined by (4.10) with u replaced by v has $\mathrm{Ric}(\widetilde{g}) = 0$ to infinite order by (4.11). We will call such a $\widetilde{g}$ an infinite-order ambient metric with log terms for $(S^1 \times \mathcal{M}, [g])$. These are parametrized by the scalar ambiguity a.

Proposition 4.9. *Let $\mathcal{M} \subset \mathbb{C}^n$ be a nondegenerate hypersurface with associated Fefferman conformal structure $(S^1 \times \mathcal{M}, [g])$ and parallel tractor 2-form $\chi \in \Gamma(\Lambda^2\mathcal{T}^*)$. For each infinite-order ambient metric $\widetilde{g}$ with log terms parametrized by $a \in C^\infty(\mathcal{M})$ as above, χ has a parallel ambient extension.*

Proof. Let $\widetilde{\mathbb{J}}$ denote the almost complex structure on $\mathbb{C}^* \times \mathbb{C}^n$. If $\widetilde{g}$ is any one of the infinite-order ambient metrics with log terms, then $\widetilde{g}_{IK}\widetilde{\mathbb{J}}^K{}_J$ is skew and $\widetilde{\nabla}\widetilde{\mathbb{J}} = 0$ since $\widetilde{g}$ is Kähler. Now $\widetilde{\mathbb{J}}$ is invariant under the dilations, so $\widetilde{\mathbb{J}}|_{\mathcal{G}}$ defines an almost complex structure on $\mathcal{T}$ (recall that $\mathcal{G}$ is identified with $\mathbb{C}^* \times \mathcal{M}$) which we denote by $\mathbb{J}$. We claim that $\mathbb{J}$ is the parallel almost complex structure of [Leit3], [ČG2]. Restricting the properties for $\widetilde{\mathbb{J}}$ shows that $h_{IK}\mathbb{J}^K{}_J$ is skew and $\mathbb{J}$ is parallel. If we write $z^0 = re^{i\theta}$ in polar coordinates, then $\mathbb{J}(r\partial_r) = \partial_\theta$. But $r\partial_r = T$ is the infinitesimal dilation and ∂_θ the conformal Killing field giving the infinitesimal S^1 action, so this is the required condition on $\mathbb{J}$. It follows that $\mathbb{J}$ is the almost complex structure of [Leit3], [ČG2]. Lowering an index, we conclude that $\widetilde{\chi}_{IJ} = \widetilde{g}_{IK}\widetilde{\mathbb{J}}^K{}_J$ is a parallel extension of $\chi_{IJ} = h_{IK}\mathbb{J}^K{}_J$. □

Note that the Einstein condition is not used in the proof of Proposition 4.9; what matters is that $\widetilde{g}$ is Kähler. In particular, Proposition 4.9 holds for all metrics of the form (4.10) so long as $J(u) = 1 + O(u^2)$. But if $\mathcal{M}$, u and a are real-analytic, then the series (4.12) converges ([Ki]) and thus defines a real-analytic function off $\mathcal{M}$. The corresponding metric $\widetilde{g}$ is then Ricci-flat and Kähler. The $(n+1,0)$-form $(z^0)^n dz^0 \wedge dz^1 \wedge \cdots \wedge dz^n$ is parallel, so $\mathrm{Hol}(\widetilde{g}) \subset SU(r+1, s+1)$. For any choice of real-analytic a, the almost complex structure and the complex volume form have thus been simultaneously extended to be parallel. In this regard it is interesting to recall that a simply connected conformal structure with conformal holonomy contained in $U(p,q)$ must have holonomy contained in $SU(p,q)$ ([Leit3]), but this is not the case

for metric holonomy. Thus for conformal structures the existence of a (local) parallel complex volume form follows from the existence of a parallel $\mathbb{J}$, but the existence of a parallel extension of $\mathbb{J}$ does not imply the existence of a parallel extension of the complex volume form.

Observe the similarity between the situation in Proposition 4.9 and that for the trivial example of extending the tractor metric. The extension $\widetilde{\chi}_{IJ}$ is the Kähler form of $\widetilde{g}$ so depends on the ambient metric which has been chosen. But the extension $\widetilde{\chi}^I{}_J = \widetilde{\mathbb{J}}^I{}_J$ with an index raised is independent of this choice. Likewise the extended parallel complex volume form $(z^0)^n dz^0 \wedge dz^1 \wedge \cdots \wedge dz^n$ is independent of choice of $\widetilde{g}$.

There is a scalar CR invariant L defined to be a constant multiple of $(J(u) - 1)/u^{n+1}|_{\mathcal{M}}$ which is independent of the choice of u satisfying $J(u) = 1 + O(u^{n+1})$. The main properties of L are derived in [Lee], [Gr1], [Gr2]. Proposition 3.10 of [GH] shows that the obstruction tensor of $(S^1 \times \mathcal{M}, [g])$ is a constant multiple of $L\theta^2$, where $\theta = \frac{i}{2}(\partial u - \bar{\partial} u)|_{T\mathcal{M}}$ is the associated pseudohermitian 1-form and the tensor $L\theta^2$ on $\mathcal{M}$ is implicitly pulled back to $S^1 \times \mathcal{M}$. So $[g]$ is obstruction-flat if and only if $L = 0$. In this case Proposition 2.16 of [Gr2] shows that for all choices of $a \in C^\infty(\mathcal{M})$ the coefficients η_k for $k \geq 1$ vanish to infinite order in the expansion of v above. The corresponding $\widetilde{g}$ are therefore smooth and are infinite-order ambient metrics in our usual sense. Hence Proposition 4.9 asserts in particular that the parallel tractor 2-form χ for an obstruction-flat Fefferman conformal structure of a nondegenerate hypersurface in $\mathbb{C}^n$ has a parallel ambient extension for a family of infinite-order ambient metrics (without log terms) parametrized by $a \in C^\infty(\mathcal{M})$. Likewise, the discussion in the paragraph following Proposition 4.9 shows that if $\mathcal{M}$ is obstruction-flat and real-analytic and a is chosen to be real-analytic, then the resulting metrics $\widetilde{g}$ are real-analytic everywhere and have holonomy contained in $SU(r+1, s+1)$.

5. G_2 Holonomy

In this section we apply Theorem 1.4 to Nurowski's conformal structures associated to a generic 2-plane field on a 5-manifold. Let $\mathcal{D}$ be a distribution of rank 2 on a manifold M of dimension 5. We assume that M is connected and, for convenience, oriented (which is equivalent to the condition that $\mathcal{D}$ is oriented). (Our arguments and results all carry over to the nonorientable case upon replacing below G_2 by $\{\pm I\}G_2 \subset O(3,4)$ and P by $\{A \in \{\pm I\}G_2 : Ae_0 = \lambda e_0, \lambda > 0\}$.) Let X, Y be a local basis for the sections of $\mathcal{D}$. $\mathcal{D}$ is said to be generic if X, Y, $[X,Y]$, $[X,[X,Y]]$, $[Y,[X,Y]]$ are linearly independent at each point. In this case we denote by $\mathcal{D}^1 = \operatorname{span}\{X, Y, [X,Y]\}$ the derived rank 3 distribution. In [C], Cartan solved the equivalence problem for such distributions by canonically associating to $\mathcal{D}$ a principal bundle $\mathcal{B} \to M$ and Cartan connection ω taking values in the Lie algebra $\mathfrak{g}_2$ of G_2, the split real form of the exceptional group. The structure group of $\mathcal{B}$ is the parabolic subgroup $P \subset G_2$ fixing a null ray. In the appendix we fix our conventions, review the Cartan connection and derive two properties which will be needed in the sequel.

In [N1], Nurowski showed that there is a conformal structure on M of signature $(2,3)$ naturally determined by $\mathcal{D}$. This follows from the existence of Cartan's canonical bundle and connection and the relation between the relevant groups. One way to see this is as follows. It is a general fact that the tangent bundle TM of a manifold M with a Cartan geometry of type $(\mathfrak{g}, P)$ is associated to the adjoint representation of P on $\mathfrak{g}/\mathfrak{p}$. We claim that the adjoint action of our P on $\mathfrak{g}_2/\mathfrak{p} \cong \mathbb{R}^5$ preserves up to scale a quadratic form of signature $(2,3)$. This implies the existence of a canonical conformal structure by the associated bundle construction. Now G_2 is a subgroup of $SO(3,4)$. Let P_c be the subgroup of $SO(3,4)$ fixing the same null ray, with Lie algebra $\mathfrak{p}_c$. Then $P \subset P_c$ and $\mathfrak{so}(3,4)/\mathfrak{p}_c \cong \mathfrak{g}_2/\mathfrak{p}$. The adjoint action of P_c on $\mathfrak{so}(3,4)/\mathfrak{p}_c$ preserves up to scale a quadratic form of signature $(2,3)$; this is the reason a Cartan geometry of type $(\mathfrak{so}(3,4), P_c)$ induces a conformal structure. Since the adjoint action of P on $\mathfrak{g}_2/\mathfrak{p}$ can be regarded as the restriction of the adjoint action of P_c, it preserves the same quadratic form up to scale.

As discussed in the introduction, any generic $\mathcal{D}$ can be written locally in the form (1.2) for some smooth function F such that F_{qq} is nonvanishing, and $\mathcal{D}$ defined by (1.2) is generic for any such F. The choice $F = q^2$ gives the homogeneous model distribution. For $\mathcal{D}$ in this form, Nurowski gave a formula in [N1], [N2] for a representative metric g_F of the conformal class such that the components of g_F and g_F^{-1} are polynomials in F, the derivatives of F of orders ≤ 4, and F_{qq}^{-1}, with coefficients which are universal functions of the local coordinates. The metric g_F is flat for $F = q^2$. In [N2], he considered the case $F = q^2 + \sum_{k=0}^{6} a_k p^k + bz$ with a_k, $b \in \mathbb{R}$, and gave an explicit formula for the ambient metric $\widetilde{g}_F$ in normal form relative to g_F. For these g_F, g_ρ in (2.1) is a polynomial in ρ of degree ≤ 2. In [LN], Leistner-Nurowski showed that the holonomy of $\widetilde{g}_F$ is contained in G_2 for all values of the a_k's and b, and is equal to G_2 if one of a_3, a_4, a_5 or a_6 is nonzero. That the holonomy is contained in G_2 is proved by exhibiting explicitly a suitable parallel 3-form (or equivalently a nonisotropic parallel spinor) for $\widetilde{g}_F$.

In [HS], Hammerl-Sagerschnig characterized those conformal structures of signature $(2,3)$ which arise from a generic distribution $\mathcal{D}$ by the existence of a parallel tractor 3-form χ compatible with the tractor metric h in the sense that

$$(U \lrcorner \chi) \wedge (V \lrcorner \chi) \wedge \chi = \lambda\, h(U,V)\, dv, \qquad U, V \in \mathcal{T}_x,$$

where dv denotes the tractor volume form and λ is some positive constant. As explained in [HS], the existence of such a χ, which is all that we need here, follows easily from the realization of $\Lambda^3\mathcal{T}^*$ and its connection as associated to the $(\mathfrak{g}_2, P)$ Cartan connection. Namely, since the 3-form $\varphi \in \Lambda^3\mathbb{R}^{7*}$ defining G_2 is fixed by P, the constant function φ on $\mathcal{B}$ is P-equivariant, so determines a section χ of the associated bundle $\Lambda^3\mathcal{T}^*$. Since φ is fixed by all of G_2 and the associated covariant derivative is given in terms of the action of $\mathfrak{g}_2$ and differentiation by vector fields on $\mathcal{B}$, it follows that χ is parallel. The compatibility condition follows from the fact that h has a similar realization as associated to the quadratic form determined by φ.

Let $\mathcal{D}$ be a generic 2-plane field on M, with $\mathcal{D}$ and M real-analytic. In the following, $\widetilde{g}$ will denote a real-analytic ambient metric for Nurowski's conformal structure associated to $\mathcal{D}$, with domain a sufficiently small dilation-invariant neighborhood $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ diffeomorphic to $\mathbb{R}_+ \times M \times \mathbb{R}$.

Proof of Theorem 1.1. By the result of Hammerl-Sagerschnig, there is a nonzero parallel section $\chi \in \Gamma(\Lambda^3\mathcal{T}^*)$ compatible with the tractor metric. By Theorem 1.4, χ has an extension as a real-analytic 3-form $\widetilde{\chi}$ on some dilation-invariant neighborhood $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ which is parallel with respect to $\widetilde{g}$. $\widetilde{\chi}$ is compatible with $\widetilde{g}$ by analytic continuation, so it follows that $\mathrm{Hol}(\widetilde{\mathcal{G}}, \widetilde{g}) \subset G_2$. □

Next we discuss Cartan's basic curvature invariant of generic 2-plane fields. In Cartan's derivation this invariant arises directly from the structure equations. We describe how it can be realized as a piece of the Weyl tensor of the conformal structure.

Let $\mathcal{D}$ be a generic 2-plane field on a (connected, oriented) 5-manifold M and let $\mathcal{D}^1 = \mathrm{span}(\mathcal{D}, [\mathcal{D}, \mathcal{D}])$ be the derived 3-plane distribution. We first define an isomorphism $\tau : \mathcal{D} \to TM/\mathcal{D}^1$ invariantly up to scale. Choose a metric g in the conformal class. In the appendix it is observed that $\mathcal{D}^1 = \mathcal{D}^\perp$, where $^\perp$ denotes orthogonal complement with respect to g. Thus the map $\psi : TM/\mathcal{D}^1 \to \mathcal{D}^*$ obtained by lowering an index with respect to g and restricting to $\mathcal{D}$ is well-defined and an isomorphism. Now g defines a negative definite metric on the line bundle $\mathcal{D}^1/\mathcal{D}$, and this line bundle is trivial since $\mathcal{D}$ is orientable. Let α be a section of $(\mathcal{D}^1/\mathcal{D})^*$ with g-length-squared equal to $-\frac{3}{4}$; α is uniquely determined up to multiplication by -1. The Lie bracket induces a pointwise map $\mathcal{D} \times \mathcal{D} \to \mathcal{D}^1/\mathcal{D}$ which we can regard as a scalar-valued nondegenerate skew form $\mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by $(X, Y) \to \alpha([X, Y])$. This form induces an identification $\mu : \mathcal{D} \to \mathcal{D}^*$ defined by $\langle \mu(X), Y\rangle = \alpha([X, Y])$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. Then $\tau = \psi^{-1} \circ \mu : \mathcal{D} \to TM/\mathcal{D}^1$ is our desired isomorphism. Clearly $\langle \mu(X), X\rangle = 0$ for $X \in \mathcal{D}$. Since ψ turns the g pairing into the duality pairing, it follows that $\tau(X) \in X^\perp/\mathcal{D}^1$. Since $X^\perp/\mathcal{D}^1$ is 1-dimensional, $\tau(X)$ actually spans $X^\perp/\mathcal{D}^1$ if $X \neq 0$. Under rescaling $\widehat{g} = \Omega^2 g$, we have $\widehat{\psi} = \Omega^2\psi$ and $\widehat{\mu} = \Omega\mu$ so that $\widehat{\tau} = \Omega^{-1}\tau$. Changing the sign of α multiplies μ and τ by -1.

Let W denote the Weyl tensor of our chosen metric g, viewed as a covariant 4-tensor. Proposition 6.1 shows that $W(\cdot, \cdot, Y, Z) = 0$ if Y, $Z \in \mathcal{D}^1$. Therefore we can define a section A of $\otimes^4\mathcal{D}^*$ by

$$A(X_1, X_2, X_3, X_4) = W(\tau(X_1), X_2, \tau(X_3), X_4), \qquad X_i \in \mathcal{D}_y \tag{5.1}$$

since the right-hand side is independent of the $\mathcal{D}^1$ ambiguity in τ. Since $\widehat{W} = \Omega^2 W$ and $\widehat{\tau} = \Omega^{-1}\tau$ and the right hand side is invariant under $\tau \to -\tau$, it follows that A is an absolute invariant of the 2-plane distribution $\mathcal{D}$. Proposition 6.2 shows that $A \in \Gamma(S^4\mathcal{D}^*)$ is symmetric.

The proof of Theorem 1.2 uses the following result which follows from the arguments of Leistner-Nurowski.

Proposition 5.1. *Let $\mathcal{D}$ be a real-analytic generic 2-plane field on a connected, simply connected, real-analytic 5-manifold M. Suppose $\mathrm{Hol}(\widetilde{g})$ is strictly contained in G_2. Then at least one of the following three conditions holds:*

(1) *$\widetilde{g}$ is locally symmetric.*
(2) *There is an open dense set $\mathcal{U} \subset M$ such that every point of $\mathcal{U}$ has a neighborhood on which there is an Einstein metric in the conformal class $[g]$.*
(3) *Every point of M has a neighborhood on which there is a metric g in the conformal class and a null line bundle $L \subset TM$ such that L is parallel for g and $Z \lrcorner \mathrm{Ric}_g = 0$ for all $Z \in L^\perp$. In this case, $W(U, K, K, Z) = 0$ for all $K \in L$, $U \in TM$, $Z \in L^\perp$.*

We give a brief outline of the proof. See [LN] for details. Berger's list contains all irreducibly acting holonomy groups of simply connected non-locally-symmetric pseudo-Riemannian manifolds. The only groups on the list in signature $(3, 4)$ are G_2 and $SO(3, 4)$. So if the holonomy is strictly contained in G_2 and $\widetilde{g}$ is not locally symmetric, its holonomy must act reducibly. If $V \subset \mathbb{R}^7$ is a nontrivial invariant subspace for the holonomy group, its orthogonal $V^\perp$ is also invariant. If $V \cap V^\perp = \{0\}$, the deRham decomposition theorem gives a local splitting of $\widetilde{g}$ as a product metric. This leads to condition (2). If V and $V^\perp$ intersect nontrivially, their intersection determines a parallel totally null distribution of rank at most 3 on the ambient space. The cases where $\dim(V \cap V^\perp) = 1$ or 3 lead to condition (2). The case where $\dim(V \cap V^\perp) = 2$ leads to condition (3). (These deductions to conditions (2) and (3) are substantial theorems in themselves.)

As regards condition (3) of Proposition 5.1, we have the following lemma.

Lemma 5.2. *Let $\mathcal{D}$ be a generic 2-plane field on a 5-manifold M and let W be the Weyl tensor at $y \in M$ of a representative of Nurowski's associated conformal structure. There exists a nonzero null vector $K \in T_yM$ such that $W(U, K, K, Z) = 0$ for all $U \in T_yM$, $Z \in K^\perp$ if and only if A_y is 3-degenerate.*

Proof. Suppose first that A_y is 3-degenerate. So there exists $0 \neq X \in \mathcal{D}$ such that $A(Y, X, X, X) = 0$ for all $Y \in \mathcal{D}$. (From now on we suppress writing $_y$.) Just take $K = X$. Certainly X is nonzero and null. In order to check $W(U, X, X, Z) = 0$, by Proposition 6.1 only the equivalence classes of U, $Z \mod \mathcal{D}^1$ are relevant. We can write $U + \mathcal{D}^1 = \tau(Y)$ for some $Y \in \mathcal{D}$, and modulo a multiplicative constant can write $Z + \mathcal{D}^1 = \tau(X)$. Then $W(U, X, X, Z) = W(\tau(Y), X, X, \tau(X)) = -A(Y, X, X, X) = 0$.

For the converse, suppose that K is nonzero and null and $W(U, K, K, Z) = 0$ for all $U \in TM$, $Z \in K^\perp$. First suppose $K \in \mathcal{D}$. In this case we take $X = K$, $Z + \mathcal{D}^1 = \tau(X)$, $U + \mathcal{D}^1 = \tau(Y)$ as above to deduce that $A(Y, X, X, X) = 0$ for all $Y \in \mathcal{D}$. There are no null vectors in $\mathcal{D}^1 \setminus \mathcal{D}$, so the only other possibility is $K \in TM \setminus \mathcal{D}^1$. In this case we can write $K + \mathcal{D}^1 = \tau(X)$ for some nonzero $X \in \mathcal{D}$. Take $Z = X$ and U to be an arbitrary vector $Y \in \mathcal{D}$ to deduce that $A(Y, X, X, X) = 0$ for all $Y \in \mathcal{D}$. □

Proof of Theorem 1.2. Theorem 1.1 shows that $\mathrm{Hol}(\widetilde{g}) \subset G_2$. So it suffices to show that the restriction of $\widetilde{g}$ to some connected open subset of its domain has holonomy equal to G_2. We can choose a connected, simply connected subset of M containing x and y. By replacing M by this subset, we may as well assume that M is simply connected. Thus we can apply Proposition 5.1. We show that injectivity of L_x and 3-nondegeneracy of A_y is incompatible with each of conditions (1)-(3) of Proposition 5.1. Note that L is injective on an open set about x since injectivity is invariant under perturbation.

At a point where A is 3-nondegenerate, it is in particular nonzero, so the Weyl tensor W of a representative metric g is nonzero. This is enough to conclude that $\widetilde{\nabla}\widetilde{R} \neq 0$ so that $\widetilde{g}$ is not locally symmetric. In fact, the second equation of (3.1) with $r = 1$ implies $\widetilde{\nabla}_T\widetilde{R} = -2\widetilde{R}$, and one has $\widetilde{R}_{ijkl} = t^2 W_{ijkl}$. So if $W \neq 0$, then $\widetilde{R} \neq 0$, so $\widetilde{\nabla}\widetilde{R} \neq 0$.

An Einstein metric has vanishing Cotton tensor. By the transformation law of the Cotton tensor, it follows that if $\widehat{g} = e^{2\omega} g$ is Einstein, then $W_{ijkl}\omega^i + C_{jkl} = 0$. Hence if (2) holds, then L is not injective on $\mathcal{U}$. Since $\mathcal{U}$ is dense, this is incompatible with injectivity of L on an open set.

If (3) holds, then Lemma 5.2 shows that A is 3-degenerate at all points of M. This violates the assumption that A_y is 3-nondegenerate. $\square$

Proof of Proposition 1.3. Represent L at the origin as a matrix in some basis. Its rank is less than 6 if and only if the determinant of each of its 6×6 submatrices vanishes. This clearly defines an algebraic subvariety in the space of jets; we have only to show that it is proper. In the appendix of [LN], Leistner-Nurowski give explicit formulae for the Weyl and Cotton tensors for the 8-parameter family $F = q^2 + \sum_{k=0}^{6} a_k p^k + bz$. It is straightforward to check from their formulae that if $a_3 \neq 0$ and $a_4 \neq 0$, then L at the origin is injective. (For instance, suppose that the right hand side of our (1.1) vanishes for some (v, λ). Taking successively $jkl = 415, 115, 413, 113, 414, 114$ shows the vanishing of, resp., v^1, v^4, λ, v^3, v^2, v^5.) So the subvariety where $\mathrm{rank}(L) < 6$ is proper.

Now A at the origin is a symmetric 4-form on $\mathcal{D}_0 = \mathrm{span}\{\partial_q, \partial_x\}$. If we represent $X \in \mathcal{D}_0$ in terms of the coordinates (u, v) dual to this basis, then the homogeneous polynomial defined by A takes the form

$$A(X, X, X, X) = A_0 u^4 + 4A_1 u^3 v + 6A_2 u^2 v^2 + 4A_3 u v^3 + A_4 v^4$$

for $A_0, \ldots, A_4 \in \mathbb{R}$ (cf. (6.6)). The conditions $A(\partial_q, X, X, X) = 0$, $A(\partial_x, X, X, X) = 0$ are

$$\begin{aligned} A_0 u^3 + 3A_1 u^2 v + 3A_2 u v^2 + A_3 v^3 &= 0 \\ A_1 u^3 + 3A_2 u^2 v + 3A_3 u v^2 + A_4 v^3 &= 0. \end{aligned} \tag{5.2}$$

So A is 3-degenerate if and only if this pair of equations has a common solution $(u, v) \in \mathbb{R}^2 \setminus (0,0)$. The set of $A_0, \dots, A_4$ such that this holds is contained in the set where the two equations have a common solution in $\mathbb{C}^2 \setminus (0,0)$, which is characterized by the vanishing of the resultant:

$$\begin{vmatrix} A_0 & 3A_1 & 3A_2 & A_3 & 0 & 0 \\ 0 & A_0 & 3A_1 & 3A_2 & A_3 & 0 \\ 0 & 0 & A_0 & 3A_1 & 3A_2 & A_3 \\ A_1 & 3A_2 & 3A_3 & A_4 & 0 & 0 \\ 0 & A_1 & 3A_2 & 3A_3 & A_4 & 0 \\ 0 & 0 & A_1 & 3A_2 & 3A_3 & A_4 \end{vmatrix} = 0. \tag{5.3}$$

This equation defines an algebraic subvariety in the space of 7-jets of F at the origin, and we must show it is proper. We cannot do this by considering Leistner-Nurowski's family of examples, because inspection of the formulae in the appendix of [LN] shows that the A which arise from their family are everywhere 2-degenerate, i.e. for any F in their 8-parameter family, at each point there is $0 \neq X \in \mathcal{D}$ such that $A(Y_1, Y_2, X, X) = 0$ for all Y_1, $Y_2 \in \mathcal{D}$. Instead we argue as follows.

It is easily seen that (5.3) defines a proper subvariety in $S^4(\mathcal{D}_0^*)$ represented as the space of A's. To see that it defines a proper subvariety in the space of 7-jets of F, it certainly suffices to show that the map from jets of F at the origin to $S^4(\mathcal{D}_0^*)$ is surjective. Take $F = q^2 + f$, where $f = f(x, y, z, p, q) = O(|(x, y, z, p, q)|^6)$. For such f, we claim that A can be identified with a nonzero constant multiple of $\left(\nabla^4 \partial_q^2 f\right)(0)|_{\mathcal{D}_0}$. Here $\partial_q^2 f$ vanishes to order 4 at the origin, $\left(\nabla^4 \partial_q^2 f\right)(0)$ denotes the symmetric 4-tensor on T_0M defined by the order 4 Taylor polynomial of $\partial_q^2 f$ at the origin, and $|_{\mathcal{D}_0}$ its restriction to $\mathcal{D}_0$. (Recall that if a smooth function φ on a manifold M vanishes to order k at a point y, then $\nabla^k \varphi(y)$ is an invariantly defined symmetric k-form on T_yM depending only on the smooth structure.) Up to an overall nonzero constant multiple, the A_i above are given by

$$A_0 = \partial_q^6 f(0),\ A_1 = \partial_q^5 \partial_x f(0),\ A_2 = \partial_q^4 \partial_x^2 f(0),\ A_3 = \partial_q^3 \partial_x^3 f(0),\ A_4 = \partial_q^2 \partial_x^4 f(0). \tag{5.4}$$

This follows by direct calculation. Equation (1.3) in [N2] gives a formula for a representative of the conformal structure in terms of F. All terms in the formula involve at most 4 derivatives of F. For $F = q^2 + f$ as above, the only terms which can contribute to the value of the curvature tensor at the origin must involve 4 derivatives of F. Inspecting term by term shows that the curvature at the origin of g_F is the same as the curvature at the origin of the metric $g_{q^2} + h$, where

$$h = -24 \partial_x^2 \partial_q^2 f \, dy^2 + 24 \partial_x \partial_q^3 f \, dy dz - 6 \partial_q^4 f \, dz^2.$$

Since g_{q^2} is flat, the curvature tensor at the origin of $g_{q^2} + h$ is given by

$$R_{ijkl} = \tfrac{1}{2} \left(h_{il,jk} - h_{jl,ik} - h_{ik,jl} + h_{jk,il} \right),$$

where the indices correspond to components and derivatives with respect to the frame $\{\partial_y, \partial_z, \partial_p, \partial_q, \partial_x\}$. Using the fact that $g_{q^2}(0) = 480dydq + 240dzdx - 320dp^2$, it is straightforward but tedious to identify the isomorphism τ appearing in (5.1), to calculate the relevant components of the Weyl tensor at the origin, and then to verify (5.4). $\square$

6. Appendix

In this appendix we collect facts about Cartan's connection [C] associated to generic 2-plane fields in the form given by Nurowski [N1] (modulo some relabeling). Other discussions may be found in the literature.

Define $\varphi \in \Lambda^3\mathbb{R}^{7*}$ by

$$\varphi = 6dx^{012} + \sqrt{3}\left(dx^{234} - dx^{135} + dx^{036}\right) + dx^{456}$$

where the coordinates are labeled $(x^0, x^1, \cdots, x^6)$ and $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$. Define $G_2 = \{A \in GL(7,\mathbb{R}) : A^*\varphi = \varphi\}$. Then $G_2 \subset SO(\widetilde{h})$, where

$$\widetilde{h}_{IJ} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h_{ij} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$h_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}_2 \subset \mathfrak{so}(\widetilde{h})$ is the set of matrices of the form

$$\begin{pmatrix} -(a_1+a_4) & a_8 & a_9 & -\frac{1}{\sqrt{3}}a_7 & \frac{1}{2\sqrt{3}}a_5 & \frac{1}{2\sqrt{3}}a_6 & 0 \\ b^1 & a_1 & a_2 & \frac{1}{\sqrt{3}}b^4 & -\frac{1}{2\sqrt{3}}b^3 & 0 & \frac{1}{2\sqrt{3}}a_6 \\ b^2 & a_3 & a_4 & \frac{1}{\sqrt{3}}b^5 & 0 & -\frac{1}{2\sqrt{3}}b^3 & -\frac{1}{2\sqrt{3}}a_5 \\ b^3 & a_5 & a_6 & 0 & \frac{1}{\sqrt{3}}b^5 & -\frac{1}{\sqrt{3}}b^4 & -\frac{1}{\sqrt{3}}a_7 \\ b^4 & a_7 & 0 & a_6 & -a_4 & a_2 & -a_9 \\ b^5 & 0 & a_7 & -a_5 & a_3 & -a_1 & a_8 \\ 0 & b^5 & -b^4 & b^3 & -b^2 & b^1 & a_1+a_4 \end{pmatrix}.$$

Define $P = \{A \in G_2 : Ae_0 = \lambda e_0, \lambda > 0\}$. Its Lie algebra $\mathfrak{p} \subset \mathfrak{g}_2$ is the subset given by $b^1 = \cdots = b^5 = 0$. The quadratic form $h_{ij}b^ib^j = -2b^1b^5 + 2b^2b^4 - (b^3)^2$ on $\mathfrak{g}_2/\mathfrak{p}$ is preserved up to scale by the adjoint action of P.

Let $\mathcal{D} \subset TM$ be a generic 2-plane field on a connected, oriented 5-manifold M. There is a principal bundle $\mathcal{B} \to M$ with structure group P and Cartan connection

$\omega : T\mathcal{B} \to \mathfrak{g}_2$ canonically associated to $\mathcal{D}$ up to equivalence. The Cartan connection can be written

$$\omega = \begin{pmatrix} -(\varphi_1+\varphi_4) & \varphi_8 & \varphi_9 & -\frac{1}{\sqrt{3}}\varphi_7 & \frac{1}{2\sqrt{3}}\varphi_5 & \frac{1}{2\sqrt{3}}\varphi_6 & 0 \\ \theta^1 & \varphi_1 & \varphi_2 & \frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{2\sqrt{3}}\theta^3 & 0 & \frac{1}{2\sqrt{3}}\varphi_6 \\ \theta^2 & \varphi_3 & \varphi_4 & \frac{1}{\sqrt{3}}\theta^5 & 0 & -\frac{1}{2\sqrt{3}}\theta^3 & -\frac{1}{2\sqrt{3}}\varphi_5 \\ \theta^3 & \varphi_5 & \varphi_6 & 0 & \frac{1}{\sqrt{3}}\theta^5 & -\frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{\sqrt{3}}\varphi_7 \\ \theta^4 & \varphi_7 & 0 & \varphi_6 & -\varphi_4 & \varphi_2 & -\varphi_9 \\ \theta^5 & 0 & \varphi_7 & -\varphi_5 & \varphi_3 & -\varphi_1 & \varphi_8 \\ 0 & \theta^5 & -\theta^4 & \theta^3 & -\theta^2 & \theta^1 & \varphi_1+\varphi_4 \end{pmatrix}$$

where the θ^i and φ_j are scalar 1-forms on $\mathcal{B}$. If σ is a local section of $\mathcal{B} \to M$, set $\bar{\theta}^i = \sigma^*\theta^i$. Then $\{\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3, \bar{\theta}^4, \bar{\theta}^5\}$ is a frame for T^*M for which $\mathcal{D} = \ker\{\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3\}$ and the derived distribution $\mathcal{D}^1 = \operatorname{span}(\mathcal{D}, [\mathcal{D}, \mathcal{D}])$ is given by $\mathcal{D}^1 = \ker\{\bar{\theta}^1, \bar{\theta}^2\}$. The metric $g = h_{ij}\bar{\theta}^i\bar{\theta}^j = -2\bar{\theta}^1\bar{\theta}^5 + 2\bar{\theta}^2\bar{\theta}^4 - (\bar{\theta}^3)^2$ is a representative for Nurowski's conformal structure. It is clear that $\mathcal{D}^\perp = \mathcal{D}^1$ with respect to such a metric.

The curvature $\Omega = d\omega + \omega \wedge \omega$ has the form

$$\Omega = \begin{pmatrix} 0 & \Phi_8 & \Phi_9 & \frac{1}{\sqrt{3}}\Phi_7 & \frac{1}{2\sqrt{3}}\Phi_5 & \frac{1}{2\sqrt{3}}\Phi_6 & 0 \\ 0 & \Phi_1 & \Phi_2 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}}\Phi_6 \\ 0 & -\Phi_3 & -\Phi_1 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}}\Phi_5 \\ 0 & \Phi_5 & \Phi_6 & 0 & 0 & 0 & \frac{1}{\sqrt{3}}\Phi_7 \\ 0 & -\Phi_7 & 0 & \Phi_6 & \Phi_1 & \Phi_2 & -\Phi_9 \\ 0 & 0 & -\Phi_7 & -\Phi_5 & -\Phi_3 & -\Phi_1 & \Phi_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.1}$$

where

$$\begin{aligned}
\Phi_1 &= C_1\theta^1 \wedge \theta^2 + B_1\theta^1 \wedge \theta^3 + B_2\theta^2 \wedge \theta^3 + A_1\theta^1 \wedge \theta^4 \\
&\qquad + A_2\theta^1 \wedge \theta^5 + A_2\theta^2 \wedge \theta^4 + A_3\theta^2 \wedge \theta^5 \\
\Phi_2 &= C_2\theta^1 \wedge \theta^2 + B_2\theta^1 \wedge \theta^3 + B_3\theta^2 \wedge \theta^3 + A_2\theta^1 \wedge \theta^4 \\
&\qquad + A_3\theta^1 \wedge \theta^5 + A_3\theta^2 \wedge \theta^4 + A_4\theta^2 \wedge \theta^5 \\
\Phi_3 &= C_0\theta^1 \wedge \theta^2 + B_0\theta^1 \wedge \theta^3 + B_1\theta^2 \wedge \theta^3 + A_0\theta^1 \wedge \theta^4 \\
&\qquad + A_1\theta^1 \wedge \theta^5 + A_1\theta^2 \wedge \theta^4 + A_2\theta^2 \wedge \theta^5 \\
\Phi_5 &= D_0\theta^1 \wedge \theta^2 + 2C_0\theta^1 \wedge \theta^3 + 2C_1\theta^2 \wedge \theta^3 + B_0\theta^1 \wedge \theta^4 \\
&\qquad + B_1\theta^1 \wedge \theta^5 + B_1\theta^2 \wedge \theta^4 + B_2\theta^2 \wedge \theta^5 \\
\Phi_6 &= D_1\theta^1 \wedge \theta^2 + 2C_1\theta^1 \wedge \theta^3 + 2C_2\theta^2 \wedge \theta^3 + B_1\theta^1 \wedge \theta^4 \\
&\qquad + B_2\theta^1 \wedge \theta^5 + B_2\theta^2 \wedge \theta^4 + B_3\theta^2 \wedge \theta^5 \\
\Phi_7 &= E\theta^1 \wedge \theta^2 + D_0\theta^1 \wedge \theta^3 + D_1\theta^2 \wedge \theta^3 + C_0\theta^1 \wedge \theta^4 \\
&\qquad + C_1\theta^1 \wedge \theta^5 + C_1\theta^2 \wedge \theta^4 + C_2\theta^2 \wedge \theta^5.
\end{aligned} \tag{6.2}$$

The coefficients A_0, A_1, A_2, A_3, A_4, B_0, B_1, B_2, B_3, C_0, C_1, C_2, D_0, D_1, E are Cartan's curvature quantities. There are further formulae for Φ_8, Φ_9 which we will not need here; see Nurowski.

The Cartan geometry $(\mathcal{B}, \omega)$ may be regarded as the reduction of a Cartan geometry $(\mathcal{B}_c, \omega_c)$ of type $(\mathfrak{so}(\widetilde{h}), P_c)$, where $P_c = \{A \in SO(\widetilde{h}) : Ae_0 = \lambda e_0, \lambda > 0\}$. Observe that (6.1) may be written

$$\Omega = \begin{pmatrix} 0 & \Omega_j & 0 \\ 0 & \Omega^i{}_j & -\Omega^i \\ 0 & 0 & 0 \end{pmatrix} \tag{6.3}$$

where

$$\Omega^i{}_j = \begin{pmatrix} \Phi_1 & \Phi_2 & 0 & 0 & 0 \\ -\Phi_3 & -\Phi_1 & 0 & 0 & 0 \\ \Phi_5 & \Phi_6 & 0 & 0 & 0 \\ -\Phi_7 & 0 & \Phi_6 & \Phi_1 & \Phi_2 \\ 0 & -\Phi_7 & -\Phi_5 & -\Phi_3 & -\Phi_1 \end{pmatrix},$$

$$\Omega_j = \begin{pmatrix} \Phi_8 & \Phi_9 & \frac{1}{\sqrt{3}}\Phi_7 & \frac{1}{2\sqrt{3}}\Phi_5 & \frac{1}{2\sqrt{3}}\Phi_6 \end{pmatrix},$$

and $\Omega^i = h^{ij}\Omega_j$. One checks directly from (6.2) that the coefficients $\Omega^i{}_{jkl}$ defined by

$$\Omega^i{}_j = \tfrac{1}{2}\Omega^i{}_{jkl}\theta^k \wedge \theta^l \qquad\qquad \Omega^i{}_{jkl} = -\Omega^i{}_{jlk}$$

satisfy $\Omega^i{}_{jil} = 0$. It follows (see, e.g., [Ko]) that ω_c is the normal Cartan connection for Nurowski's conformal structure. Lowering an index gives $\Omega_{ij} = \frac{1}{2}\Omega_{ijkl}\theta^k \wedge \theta^l$:

$$\Omega_{ij} = \begin{pmatrix} 0 & \Phi_7 & \Phi_5 & \Phi_3 & \Phi_1 \\ -\Phi_7 & 0 & \Phi_6 & \Phi_1 & \Phi_2 \\ -\Phi_5 & -\Phi_6 & 0 & 0 & 0 \\ -\Phi_3 & -\Phi_1 & 0 & 0 & 0 \\ -\Phi_1 & -\Phi_2 & 0 & 0 & 0 \end{pmatrix}. \tag{6.4}$$

(This corrects the corresponding formula given in the appendix of [N1].) From (6.2) and (6.4) one reads off the following:

$$\begin{aligned} A_0 &= \Omega_{1414} \\ A_1 &= \Omega_{1415} = \Omega_{1424} = \Omega_{1514} = \Omega_{2414} \\ A_2 &= \Omega_{1425} = \Omega_{1515} = \Omega_{1524} = \Omega_{2415} = \Omega_{2424} = \Omega_{2514} \\ A_3 &= \Omega_{1525} = \Omega_{2425} = \Omega_{2515} = \Omega_{2524} \\ A_4 &= \Omega_{2525}. \end{aligned} \tag{6.5}$$

If σ is a section of $\mathcal{B} \to M$ as above, then comparing (6.3) with the form of the curvature of the normal conformal Cartan connection shows that $W_{ijkl} = \Omega_{ijkl} \circ \sigma$ are the components of the Weyl tensor of the metric $g = h_{ij}\bar{\theta}^i\bar{\theta}^j$ in the frame $\{\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3, \bar{\theta}^4, \bar{\theta}^5\}$.

Similarly, setting $\Omega_j = \frac{1}{2}\Omega_{jkl}\theta^k \wedge \theta^l$ with $\Omega_{jkl} = -\Omega_{jlk}$, the components of the Cotton tensor are given by $C_{jkl} = \Omega_{jkl} \circ \sigma$. Thus this expresses the Weyl and Cotton curvature in terms of Cartan's scalar invariants.

Proposition 6.1. *$W(\cdot,\cdot,Y,Z) = 0$ if Y, $Z \in \mathcal{D}^1$.*

Proof. This is clear since $\Omega_{ijkl} = 0$ if i, $j \geq 3$ from (6.4). (Or observe from (6.2) that $\Omega_{ij} = 0 \mod \theta^1, \theta^2$ for all i, j.) □

Proposition 6.2. *The tensor A defined in* (5.1) *is symmetric, i.e. $A \in \Gamma(S^4\mathcal{D}^*)$.*

Proof. Consider the maps ψ, μ, τ defined in the paragraph before (5.1). Let again $\{\bar\theta^1, \bar\theta^2, \bar\theta^3, \bar\theta^4, \bar\theta^5\}$ be a coframe on M obtained by pulling back by a local section of the principal bundle and let $\{U_1, U_2, U_3, U_4, U_5\}$ be the dual frame. Then $\mathcal{D} = \operatorname{span}\{U_4, U_5\}$ and $\mathcal{D}^1 = \operatorname{span}\{U_3, U_4, U_5\}$. Lowering an index shows that

$$\psi(U_1 + \mathcal{D}^1) = -\bar\theta^5|_{\mathcal{D}}, \qquad \psi(U_2 + \mathcal{D}^1) = \bar\theta^4|_{\mathcal{D}}.$$

Taking $\alpha = \frac{\sqrt{3}}{2}\,\bar\theta^3$, consideration of the ${}^0{}_3$ component of $\Omega = d\omega + \omega \wedge \omega$ shows that

$$\mu(U_4) = -\bar\theta^5|_{\mathcal{D}}, \qquad \mu(U_5) = \bar\theta^4|_{\mathcal{D}}.$$

Thus

$$\tau(U_4) = U_1 + \mathcal{D}^1, \qquad \tau(U_5) = U_2 + \mathcal{D}^1.$$

Hence (6.5) shows that A defined by (5.1) is symmetric. □

Note that (6.5) in fact shows that

$$A = A_0u^4 + 4A_1u^3v + 6A_2u^2v^2 + 4A_3uv^3 + A_4v^4, \tag{6.6}$$

where $u = \bar\theta^4|_{\mathcal{D}}$, $v = \bar\theta^5|_{\mathcal{D}}$, and we abuse notation by denoting also by A_0, A_1, A_2, A_3, A_4 their pullbacks under the local section σ of $\mathcal{B}$.

The reductive part of P is $\mathbb{R}_+ \cdot SL(2,\mathbb{R})$, where $SL(2,\mathbb{R})$ is viewed as a subgroup of $SO(h)$ via

$$SL(2,\mathbb{R}) \ni S \mapsto \begin{pmatrix} S & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S \end{pmatrix} \in SO(h).$$

The space of Weyl tensors is a 35-dimensional irreducible representation for $SO(h)$. Upon restriction to $SL(2,\mathbb{R})$ it decomposes as $S^4 \oplus 2S^3 \oplus 3S^2 \oplus 4S^1 \oplus 5\mathbb{R}$, where S^k denotes the k^{th} symmetric power of the standard representation and $\mathbb{R} = S^0$ the trivial representation. The space of Weyl tensors which arise from a conformal structure associated to a generic 2-plane distribution is the 15-dimensional subspace given by (6.2), (6.4). This decomposes under $SL(2,\mathbb{R})$ as $S^4 \oplus S^3 \oplus S^2 \oplus S^1 \oplus \mathbb{R}$ corresponding to the division of Cartan's scalar invariants into A's, B's, C's, D's, and E.

References

[BEG] T. N. Bailey, M. G. Eastwood and A. R. Gover, *Thomas's structure bundle for conformal, projective and related structures*, Rocky Mountain J. Math. **24** (1994), 1191–1217.

[B] O. Biquard, *Continuation unique à partir de l'infini conforme pour les métriques d'Einstein*, Math. Res. Lett. **15** (2008), 1091–1099. `arXiv:0708.4346`.

[BG] T. Branson and A. R. Gover, *Conformally invariant operators, differential forms, cohomology and a generalisation of Q-curvature*, Comm. P. D. E. **30** (2005), 1611–1669, `arXiv:math/0309085`.

[Br1] R. Bryant, *Metrics with exceptional holonomy*, Ann. Math. **126** (1987), 525–576.

[Br2] R. Bryant, *Conformal geometry and 3-plane fields on 6-manifolds*, Developments of Cartan Geometry and Related Mathematical Problems, RIMS Symposium Proceedings (Kyoto University) **1502**, 2006, 1–15, `arXiv:math/0511110`.

[BH] R. L. Bryant and L. Hsu, *Rigidity of integral curves of rank 2 distributions*, Invent. Math. **114** (1993), 435–461.

[ČG1] A. Čap and A. R. Gover, *Standard tractors and the conformal ambient metric construction*, Ann. Global Anal. Geom. **24** (2003), 231–259, `arXiv:math/0207016`.

[ČG2] A. Čap and A. R. Gover, *A holonomy characterisation of Fefferman spaces*, Ann. Global Anal. Geom. **38** (2010), 399–412, `arXiv:math/0611939`.

[C] É. Cartan, *Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. École Norm. Sup. **27** (1910), 109–192.

[F] C. Fefferman, *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains*, Ann. Math. **103** (1976), 395–416; Correction: Ann. Math. **104** (1976), 393–394.

[FG1] C. Fefferman and C. R. Graham, *Conformal invariants,* in *The Mathematical Heritage of Élie Cartan (Lyon, 1984)*, Astérisque, 1985, Numero Hors Serie, 95–116.

[FG2] C. Fefferman and C. R. Graham, *The Ambient Metric*, to appear, Princeton University Press, `arXiv:math/0710.0919`.

[Go1] A. R. Gover, *Almost conformally Einstein manifolds and obstructions*, Differential geometry and its applications, Proceedings of the 9th International Conference (DGA 2004), Matfyzpress, Prague, 2005, 247–260, `arXiv:math/0412393`.

[Go2] A. R. Gover, *Laplacian operators and Q-curvature on conformally Einstein manifolds*, Math. Ann. **336** (2006), 311–334, `arXiv:math/0506037`.

[Go3] A. R. Gover, *Almost Einstein and Poincaré-Einstein manifolds in Riemannian signature*, J. Geom. Phys. **60** (2010), 182–204, `arXiv:0803.3510`.

[Gr1] C.R. Graham, *Scalar boundary invariants and the Bergman kernel*, Complex Analysis II, Proceedings, Univ. of Maryland 1985-86, Springer Lecture Notes **1276**, 108–135.

[Gr2] C.R. Graham, *Higher asymptotics of the complex Monge-Ampère equation*, Comp. Math. **64** (1987), 133–155.

[GH] C. R. Graham and K. Hirachi, *Inhomogeneous ambient metrics*, Symmetries and Overdetermined Systems of Partial Differential Equations, 403–420, IMA Vol. Math. Appl. **144**, Springer, 2008, `arXiv:math/0611931`.

[GL] C. R. Graham and J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), 186–225.

[H] M. Hammerl, *Invariant prolongation of BGG-operators in conformal geometry*, Arch. Math. (Brno) **44** (2008), 367–384, `arXiv:0811.4122`.

[HS] M. Hammerl and K. Sagerschnig, *Conformal structures associated to generic rank 2 distributions on 5-manifolds–characterization and Killing-field decomposition*, SIGMA **5** (2009), 081, 29 pages, `arXiv:0908.0483`.

[J] A. Juhl, *Explicit formulas for GJMS-operators and Q-curvatures*, `arXiv:1108.0273`.

[Ki] S. Kichenassamy, *Fuchsian Reduction*, Birkhäuser, 2007.

[Ko] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer, 1972.

[Lee] J. Lee, *Higher asymptotics of the complex Monge-Ampère equation and geometry of CR-manifolds*, MIT Ph.D. thesis, 1982.

[Leis] T. Leistner, *Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds*, Diff. Geom. Appl. **24** (2006), 458–478, `arXiv:math/0501239`.

[LN] T. Leistner and P. Nurowski, *Conformal structures with $G_{2(2)}$-ambient metrics*, `arXiv:0904.0186`.

[Leit1] F. Leitner, *Normal conformal Killing forms*, `arXiv:math/0406316`.

[Leit2] F. Leitner, *Conformal Killing forms with normalisation condition*, Rend. Circ. Mat. Palermo (2) Suppl. No. 75 (2005), 279–292.

[Leit3] F. Leitner, *A remark on unitary conformal holonomy*, Symmetries and Overdetermined Systems of Partial Differential Equations, 445–460, IMA Vol. Math. Appl. **144**, Springer, 2008, `arXiv:math/0604393`.

[N1] P. Nurowski, *Differential equations and conformal structures*, J. Geom. Phys. **55** (2005), 19–49, `arXiv:math/0406400`.

[N2] P. Nurowski, *Conformal structures with explicit ambient metrics and conformal G_2 holonomy*, Symmetries and Overdetermined Systems of Partial Differential Equations, 515–526, IMA Vol. Math. Appl. **144**, Springer, 2008, `arXiv:math/0701891`.

[T] T. Y. Thomas, *On conformal geometry*, Proc. Natl. Acad. Sci. USA **12** (1926), 352–359.

[W] T. Willse, *Parallel tractor extension and metrics of split G_2 holonomy*, University of Washington Ph.D. thesis, 2011.

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